Engineering with Computers (2016) 32:405–424
DOI 10.1007/s00366-015-0422-1

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ORIGINAL ARTICLE

Geometric validity (positive Jacobian) of high-order Lagrange finite elements, theory and practical guidance

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Received: 28 July 2014 / Accepted: 9 September 2015 / Published online: 3 October 2015

Abstract Finite elements of degree two or more are needed to solve various P.D.E. problems. This paper discusses a method to validate such meshes for the case of the usual Lagrange elements of various degrees. The first section of this paper comes back to Bézier curve and Bézier patches of arbitrary degree. The way in which a Bézier patch and a finite element are related is recalled. The usual Lagrange finite elements of various degrees are discussed, including simplices (triangle and tetrahedron), quads, prisms (pentahedron), pyramids and hexes together with some low-degree Serendipity elements. A validity condition, the positivity of the jacobian, is exhibited for these elements. Elements of various degrees are envisaged also including some "linear" elements (therefore straight-sided elements of degree 1) because the jacobian (polynomial) of some of them is not totally trivial.

Keywords High-order finite elements · Bézier curves · Bézier patches

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List of symbols

$\hat{K}$	The reference element, K the current element, F_K the mapping from $\hat{K}$ to K, $p_i, p_{ij}, \ldots$, a shape function, d the degree of the finite element, $\mathcal{J}$ the jacobian of K, q the degree of this jacobian,
$\hat{A}_i$, A_i, $(A_{ij}, A_{ijk}, A_{ijkl})$	A node of $\hat{K}$ and its image by F_K
u, v, w, t or $\hat{x}, \hat{y}, \hat{z}$	The parameters living in the parametric space, e.g. $\hat{K}$
Γ and γ, Σ and σ, Θ and θ, resp	A curve and its expression, a bidimensional patch and its expression, a tridimensional patch and its expression,
P_{ij} (P_{ijk}, P_{ijkl})	A control point, N_{ij} (N_{ijk}, N_{ijkl}) a (scalar) control value
$B_i^d(u)$	The Bernstein polynomial of degree d for a system of natural coordinates, $B_{ij}^d(u,v), B_{ijk}^d(u,v,w), \ldots$, the Bernstein polynomial of degree d for a system of barycentric coordinates, C_i^d, the binomial coefficient,
$[.]$, $\{.\}$, $\lvert.\rvert$, $(. \wedge .)$ and $<. >$	A matrix, a vector, a determinant, the cross product and the dot product.

1 Introduction

High-order (p-version) finite elements are employed to accurately solve a number of P.D.E. with a good rate of convergence, see [2,3,8,9,21]. The order impacts two different aspects, one concerning the geometry, and the other the finite element approximation. These two aspects may be combined or not. For instance, a high-order element in the case of a straight-sided geometry does not lead to any difficulty at the time the geometry is considered, while even a not too high-order element where the geometry is a curved geometry may lead to some tedious questions, see the pioneering references, [22–24] and [11]. In this paper, we are only concerned with the geometric validity of high-order meshes of planar or volume domains with curved boundaries but we are not directly interested in the finite element aspect, e.g. solution methods and mesh quality. As regards the validity of a given mesh, a common idea is that it is sufficient to locate the nodes on the curved edges without giving any explicit attention to the positivity of the resulting jacobian. Another idea and one that is advocated in a number of papers is to evaluate the jacobian on a sample of points (for example Gauss points) but this is only a necessary condition. Actually, this works well in most cases but only if the boundary is not too bended. This is why we decided to consider this problem by returning to the purely theoretical point of view with a deliberate geometric touch, as we did in [7] for tetrahedral elements, in [16] for quadrilaterals and as can also be found in [19] and, more recently, in [20] for high-order triangles.

2 Bernstein polynomials, Bézier curves and Bézier patches

Following [5] and [12], a Bézier curve of degree d is defined by means of $d+1$ control points and the Bernstein basis. More precisely, let $P_i \in R^2$ or R^3 be those points, the curve Γ reads

$$\Gamma = \left\{ \gamma(u) = \sum_{i=0,d} B_i^d(u) P_i \text{ with } u \in [0,1] \right\} \tag{1}$$

and, using a system of barycentric coordinates, the same reads

$$\Gamma = \left\{ \gamma(u,v) = \sum_{i+j=d} B_{ij}^d(u,v) P_{ij}, \text{ with } u+v=1 \right\} \tag{2}$$

In the above equations, the Bernstein polynomials, respectively, read as

$$B_i^d(u) = C_i^d u^i (1-u)^{d-i} = \frac{d!}{i!(d-i)!} u^i (1-u)^{d-i} \tag{3}$$

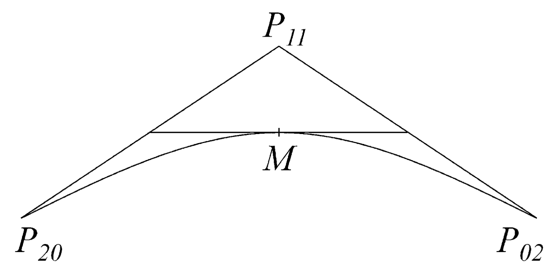

Fig. 1 The Bézier curve of degree 2 related to the control points P_{20}, P_{11} and P_{02}

and

$$B_{ij}^d(u,v) = C_{ij}^d u^i v^j = \frac{d!}{i!j!} u^i v^j \text{ with } i+j=d, \tag{4}$$

and so on.

In the case where $d=2$, Γ is an arc of parabola, Fig. 1, passing through P_{20} and P_{02}, the tangent at P_{20} is parallel to $\overrightarrow{P_{20}P_{11}}$, that at P_{02} is parallel to $\overrightarrow{P_{11}P_{02}}$. Let M be the mid-node of Γ, e.g. $M = \gamma(\frac{1}{2})$, then the tangent at M is parallel to $\overrightarrow{P_{20}P_{02}}$. Moreover, M, in terms of the P_{ij}, reads
$M = \dfrac{2P_{11} + P_{20} + P_{02}}{4}$ and, conversely, we have
$P_{11} = \dfrac{4M - P_{20} - P_{02}}{2}$.

This abstract reading of $\gamma(u)$ extends to tensor-product patches, for instance, in two dimension, and for the degree $d \times d$, we have

$$\sigma(u,v) = \sum_{i=0,d} \sum_{j=0,d} B_i^d(u) B_j^d(v) P_{ij}, \tag{5}$$

and the patch reads as

$$\Sigma = \{\sigma(u,v) \text{ with } (u,v) \in [0,1] \times [0,1]\}, \tag{6}$$

and this extends to tridimensional tensor-product patches (e.g. by defining $\theta(u,v,w), \Theta, \ldots$ accordingly). Such definitions will be used to define quadrilaterals, hexes and quadrilateral faces (in the case of a prism or a pyramid).

As for simplices or triangular faces, it is much more convenient to use the *barycentric* form of the Bézier setting, e.g.

$$\Sigma = \left\{ \sigma(u,v,w) = \sum_{i+j+k=d} B_{ijk}^d(u.v,w) P_{ijk}, \quad u+v+w=1 \right\} \tag{7}$$

for a triangle or a triangular face and

$$\Theta = \left\{ \theta(u,v,w,t) = \sum_{i+j+k+l=d} B^d_{ijkl}(u.v,w,t)P_{ijkl} \right\} \quad (8)$$

with $u+v+w+t=1$, for a tet patch.

Before going further, let us recall the rule of derivation and the rule of multiplication as they apply to Bernstein polynomials. For the derivatives, we have

$$\frac{\mathrm{d}}{\mathrm{d}u} B^d_i(u) = d\ (B^{d-1}_{i-1}(u) - B^{d-1}_i(u)), \quad (9)$$

together with (barycentric coordinates)

$$\frac{\partial}{\partial u} B^d_{ij}(u,v) = d\ B^{d-1}_{i-1,j}(u,v) \quad (10)$$

$$\frac{\partial}{\partial v} B^d_{ij}(u,v) = d\ B^{d-1}_{i,j-1}(u,v) \quad (11)$$

Then, it is easy to see that

$$\frac{\mathrm{d}}{\mathrm{d}u} \sum_{i=0,d} B^d_i(u)P_i = d \sum_{i=0,d-1} B^{d-1}_i(u)\overrightarrow{P_iP_{i+1}},$$

with the following diagram for the P_is:

$$P_0 \quad P_1 \quad P_2 \cdots P_{d-1} \quad P_d$$

and $$\frac{\partial}{\partial u} \sum_{i=0,d} \sum_{j=0,d} B^d_i(u)B^d_j(v)P_{ij}$$
$$= d \sum_{i=0,d-1} \sum_{j=0,d} B^{d-1}_i(u)B^d_j(v)\overrightarrow{P_{ij}P_{i+1,j}},$$
$$\frac{\partial}{\partial v} \sum_{i=0,d} \sum_{j=0,d} B^d_i(u)B^d_j(v)P_{ij}$$
$$= d \sum_{i=0,d} \sum_{j=0,d-1} B^d_i(u)B^{d-1}_j(v)\overrightarrow{P_{ij}P_{i,j+1}},$$

with, now, the following diagram for the P_{ij}s:

$$\begin{array}{cccccc}
P_{0d} & P_{1d} & P_{2d} & \ldots & P_{d-1,d} & P_{dd} \\
P_{0,d-1} & P_{1,d-1} & P_{2,d-1} & \ldots & P_{d-1,d-1} & P_{d,d-1} \\
 & & & \ldots & & \\
 & & & \ldots & & \\
P_{01} & P_{11} & P_{21} & \ldots & P_{d-1,1} & P_{d1} \\
P_{00} & P_{10} & P_{20} & \ldots & P_{d-1,0} & P_{d0}
\end{array}$$

etc., and (for a barycentric system)

$$\frac{\partial}{\partial u} \sum_{i+j=d} B^d_{ij}(u,v)P_{ij} = d \sum_{i+j=d-1} B^{d-1}_{ij}(u,v)P_{i+1,j},$$
$$\frac{\partial}{\partial v} \sum_{i+j=d} B^d_{ij}(u,v)P_{ij} = d \sum_{i+j=d-1} B^{d-1}_{ij}(u,v)P_{i,j+1},$$

with the following diagram for the P_{ij}s:

$$P_{d0} \quad P_{d-1,1} \quad P_{d-2,2} \quad \ldots \quad P_{1,d-1} \quad P_{0,d}$$

and

$$\frac{\partial}{\partial u} \sum_{i+j+k=d} B^d_{ijk}(u,v,w)P_{ijk}$$
$$= d \sum_{i+j+k=d-1} B^{d-1}_{ijk}(u,v,w)P_{i+1,j,k},$$
$$\frac{\partial}{\partial v} \sum_{i+j+k=d} B^d_{ijk}(u,v,w)P_{ijk}$$
$$= d \sum_{i+j+k=d-1} B^{d-1}_{ijk}(u,v,w)P_{i,j+1,k},$$
$$\frac{\partial}{\partial w} \sum_{i+j+k=d} B^d_{ijk}(u,v,w)P_{ijk}$$
$$= d \sum_{i+j+k=d-1} B^{d-1}_{ijk}(u,v,w)P_{i,j,k+1}, \quad (12)$$

with, now, the following diagram for the P_{ijk}s:

$$\begin{array}{ccccccc}
 & & & P_{0,0,d} & & & \\
 & & & \ldots & & & \\
 & & P_{d-2,0,2} & \ldots & P_{0,d-2,2} & & \\
 & P_{d-1,0,1} & P_{d-2,1,1} & \ldots & P_{1,d-2,1} & P_{0,d-1,1} & \\
P_{d,0,0} & P_{d-1,1,0} & P_{d-2,2,0} & \ldots & P_{2,d-2,0} & P_{1,d-1,0} & P_{0,d,0}
\end{array}$$

etc.

For the multiplication, we have

$$B^d_i(u)B^e_j(u) = \frac{C^d_i C^e_j}{C^{d+e}_{i+j}} B^{d+e}_{i+j}(u), \quad (13)$$

together with

$$B^d_{i_1j_1}(u,v)B^e_{i_2j_2}(u,v) = \frac{C^d_{i_1j_1} C^e_{i_2j_2}}{C^{d+e}_{i_1+i_2,j_1+j_2}} B^{d+e}_{i_1+i_2,j_1+j_2}(u,v) \quad (14)$$

in the case of a curve but also for general patches. These rules will be of great interest at the time we will compute the jacobian of the elements in the next sections.

3 Bézier form versus finite element form of an element

This section briefly recalls the basics of what a finite element is and shows that a complete[1] Lagrange finite element can be written in terms of a Bézier form. To do this, we follow [8] and more precisely [9], using the same notations.

Let K be a geometric element (triangle, quad, tet, etc.), the Lagrange finite element associated with K is defined by the triple $[K, P, Nodes]$ where K is the element, P is a set

[1] incomplete element can also be written in this way but it is more subtle.

of polynomials and *Nodes* is a set of nodes. Actually, K is constructed as the image of a reference element $\hat{K}$, equipped with a set of reference nodes, by means of a mapping F_K, e.g. $K = F_K(\hat{K})$ and, in turn, F_K is defined by means of the polynomials in P and we have $F_K(\hat{A}) = \sum_{i=0,n-1} p_i(\hat{A})A_i$, where p_i is a polynomial, n is the number of such polynomials (e.g. the dimension of space P), A_i is the node i of K and $\hat{A}$ is the value of the parameters (e.g. for instance, (u, v) or $(\hat{x}, \hat{y})$) where F_K is evaluated. Therefore, if we consider K as a patch (such as Σ in the previous section), we have (with evident notations)

$$K = \left\{ M(u,v)) = \sum_{i=0,n-1} p_i(u,v)A_i, (u,v) \in \hat{K} \right\}, \tag{15}$$

in other words, the finite element is defined by means of *shape functions* and *nodes*.

Let us consider now a Bézier form like (here is the case of a quadrilateral patch)

$$\sigma(u,v) = \sum_{i=0,d} \sum_{j=0,d} B_i^d(u)B_j^d(v)P_{ij},$$

where d is the degree of space P and P_{ij} is a set of control points, i.e. a patch defined by means of *Bernstein polynomials* and *control points*.

As a matter of fact, for a complete element, the space P is complete so that it can be expressed both in terms of the above $p_i(u,v)$ and the Bernstein polynomials which are two equivalent bases of the polynomial space. In other words, we have (with appropriate notations)

$$\sum_{i=0,n-1} p_i(u,v)A_i = \sum_{i=0,d} \sum_{j=0,d} B_i^d(u)B_j^d(v)P_{ij}.$$

As a consequence, the A_is can be written in terms of the P_{ij}s and vice versa, and the p_is are linear combinations of the B_i^d and vice versa.

To simply illustrate this point, we return to the simple case of a Bézier curve of degree 2. It reads

$$\sum_{i=0,2} B_i^2(u)P_i = (1-u)^2P_0 + 2u(1-u)P_1 + u^2P_2,$$

let us define $A_0 = P_0, A_1 = \frac{P_0+P_2+2P_1}{4}$ and $A_2 = P_2$, then $P_0 = A_0, P_1 = \frac{-A_0-A_2+4A_1}{2}$ and $P_2 = A_2$ and, replacing the P_is in the above relation, we have

$$\begin{aligned}&(1-u)^2A_0 + 2u(1-u)\frac{-A_0-A_2+4A_1}{2} + u^2A_2 \\ &= (1-u)(1-2u)A_0 + 4u(1-u)A_1 + u(2u-1)A_2,\end{aligned}$$

which is the classical Lagrange form of the curve. This mechanism applies whatever the degree of the curve and also applies for the patches themselves. The main interest is then to replace the finite elements by their equivalent Bézier setting, making simpler and systematic the calculation and the analysis of their jacobian polynomials using the related convex hull property.

4 Computing and evaluating the jacobian

First of all, we introduce the control points associated with a given element K (formulae allows for this in the case K is defined by its nodes, see Appendix). Then, we write the finite element in its Bézier setting and we express its jacobian. This polynomial being a product of Bernstein polynomials (derivatives are multiplied one each other) is, itself, a Bézier form. Therefore, we have immediately a sufficient condition of positiveness: the coefficients of the polynomial must be strictly positive in the case of an interpolant coefficient and non-negative if not.

Before giving the exhaustive catalogue of elements, we give a detailed description of the 9-node quad as an illustration.

4.1 The 9-node Lagrange quadrilateral

The geometric view of a finite element leads us to see it as a patch defined in a parametric space, here $[0,1] \times [0,1]$, the unit-square. Since the 9-node quad is a complete element of degree 2, we have

$$\sigma(u,v) = \sum_{i=0,2} \sum_{j=0,2} B_i^2(u)B_j^2(v)\,P_{i,j}, \tag{16}$$

using the derivative rule, cf. [12], we have

$$\frac{\partial\sigma(u,v)}{\partial u} = 2\sum_{i=0,1} \sum_{j=0,2} B_i^1(u)B_j^2(v)\,\Delta_{i,j}^{10}$$

and, similarly $$\frac{\partial\sigma(u,v)}{\partial v} = 2\sum_{i=0,2} \sum_{j=0,1} B_i^2(u)B_j^1(v)\,\Delta_{i,j}^{01}$$

with

$$\Delta_{i,j}^{10} = \overrightarrow{P_{ij}P_{i+1,j}} \text{ and } \Delta_{i,j}^{01} = \overrightarrow{P_{ij}P_{i,j+1}}.$$

Then, the Jacobian matrix (of the derivatives) is

$$\mathcal{M} = \left[\frac{\partial\sigma(u,v)}{\partial u}\ \frac{\partial\sigma(u,v)}{\partial v}\right], \tag{17}$$

and the jacobian (determinant of the above matrix) simply reads

$$\mathcal{J}(u,v) = 4\left| \sum_{i_1=0,1} \sum_{j_1=0,2} B_{i_1}^1(u)B_{j_1}^2(v)\Delta_{i_1,j_1}^{10} \quad \sum_{i_2=0,2} \sum_{j_2=0,1} B_{i_2}^2(u)B_{j_2}^1(v)\Delta_{i_2,j_2}^{01} \right|,$$

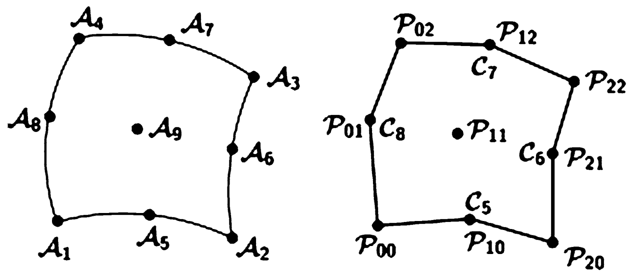

Fig. 2 Synopsis of a 9-node quad, node numbering and control point numbering

or

$$\mathcal{J}(u,v) = 4 \sum_{i_1=0,1} \sum_{j_1=0,2} \sum_{i_2=0,2} \sum_{j_2=0,1} B^1_{i_1}(u) B^2_{j_1}(v) B^2_{i_2}(u) B^1_{j_2}(v) \left| \Delta^{10}_{i_1,j_1} \quad \Delta^{01}_{i_2,j_2} \right|,$$

then using the multiplication rule, cf.[12], we have

$$B^1_i(u) B^2_k(u) = \frac{C^1_i C^2_k}{C^3_{i+k}} B^3_{i+k}(u),$$

and the same holds in v. Hence

$$\mathcal{J}(u,v) = 4 \sum_{i_1=0,1} \sum_{j_1=0,2} \sum_{i_2=0,2} \sum_{j_2=0,1} K^{1,0}_{i_1 i_2} K^{0,1}_{j_1 j_2} B^3_{i_1+i_2}(u) B^3_{j_1+j_2}(v) \left| \Delta^{10}_{i_1,j_1} \quad \Delta^{01}_{i_2,j_2} \right|,$$

with

$$K^{1,0}_{i_1 i_2} = \frac{C^1_{i_1} C^2_{i_2}}{C^3_{i_1+i_2}} = \frac{C^2_{i_2}}{C^3_{i_1+i_2}} \text{ and } K^{0,1}_{j_1 j_2} = \frac{C^2_{j_1} C^1_{j_2}}{C^3_{j_1+j_2}} = \frac{C^2_{j_1}}{C^3_{j_1+j_2}}.$$

To complete the final formula, we group together the terms to find the following generic expression:

$$\mathcal{J}(u,v) = 4 \sum_{I=0,3} \sum_{J=0,3} B^3_I(u) B^3_J(v) Q_{IJ}, \tag{18}$$

which leads to finding the coefficients Q_{IJ}. At the same time, we know the degree of the jacobian polynomial, 3 in each direction (u and v) and we see that the number of control coefficients is 16.

The final formula for the coefficients[2] is

$$Q_{IJ} = \sum_{i_1+i_2=I} \sum_{j_1+j_2=J} \frac{C^2_{i_2}}{C^3_{i_1+i_2}} \frac{C^2_{j_1}}{C^3_{j_1+j_2}} \left| \Delta^{10}_{i_1,j_1} \quad \Delta^{01}_{i_2,j_2} \right| \tag{19}$$

and, *in extenso*, after replacing the P_{ij} by the A_i and C_i (see Fig. 2 for the correspondence of these notations), we have

$$\begin{aligned}
Q_{00} &= \left|\overrightarrow{A_1C_5} \quad \overrightarrow{A_1C_8}\right| \\
Q_{01} &= \frac{1}{3}\left|\overrightarrow{A_1C_5} \quad \overrightarrow{C_8A_4}\right| + \frac{2}{3}\left|\overrightarrow{C_8C_9} \quad \overrightarrow{A_1C_8}\right| \\
Q_{02} &= \frac{1}{3}\left|\overrightarrow{A_4C_7} \quad \overrightarrow{A_1C_8}\right| + \frac{2}{3}\left|\overrightarrow{C_8C_9} \quad \overrightarrow{C_8A_4}\right| \\
Q_{03} &= \left|\overrightarrow{A_4C_7} \quad \overrightarrow{C_8A_4}\right| \\
Q_{10} &= \frac{2}{3}\left|\overrightarrow{A_1C_5} \quad \overrightarrow{C_5C_9}\right| + \frac{1}{3}\left|\overrightarrow{C_5A_2} \quad \overrightarrow{A_1C_8}\right| \\
Q_{11} &= \frac{2}{9}\left|\overrightarrow{A_1C_5} \quad \overrightarrow{C_9C_7}\right| + \frac{4}{9}\left|\overrightarrow{C_8C_9} \quad \overrightarrow{C_5C_9}\right| \\
&\quad + \frac{1}{9}\left|\overrightarrow{C_5A_2} \quad \overrightarrow{C_8A_4}\right| + \frac{2}{9}\left|\overrightarrow{C_9C_6} \quad \overrightarrow{A_1C_8}\right| \\
Q_{12} &= \frac{2}{9}\left|\overrightarrow{A_4C_7} \quad \overrightarrow{C_5C_9}\right| + \frac{4}{9}\left|\overrightarrow{C_8C_9} \quad \overrightarrow{C_9C_7}\right| \\
&\quad + \frac{1}{9}\left|\overrightarrow{C_7A_3} \quad \overrightarrow{A_1C_8}\right| + \frac{2}{9}\left|\overrightarrow{C_9C_6} \quad \overrightarrow{C_8A_4}\right| \\
Q_{13} &= \frac{2}{3}\left|\overrightarrow{A_4C_7} \quad \overrightarrow{C_9C_7}\right| + \frac{1}{3}\left|\overrightarrow{C_7A_3} \quad \overrightarrow{C_8C_4}\right| \\
Q_{20} &= \frac{1}{3}\left|\overrightarrow{A_1C_5} \quad \overrightarrow{A_2C_6}\right| + \frac{2}{3}\left|\overrightarrow{C_5A_2} \quad \overrightarrow{C_5C_9}\right| \\
Q_{21} &= \frac{1}{9}\left|\overrightarrow{A_1C_5} \quad \overrightarrow{C_6A_3}\right| + \frac{2}{9}\left|\overrightarrow{C_8C_9} \quad \overrightarrow{A_2C_6}\right| \\
&\quad + \frac{2}{9}\left|\overrightarrow{C_5A_2} \quad \overrightarrow{C_9C_7}\right| + \frac{4}{9}\left|\overrightarrow{C_9C_6} \quad \overrightarrow{C_5C_9}\right| \\
Q_{22} &= \frac{1}{9}\left|\overrightarrow{A_4C_7} \quad \overrightarrow{A_2C_6}\right| + \frac{2}{9}\left|\overrightarrow{C_8C_9} \quad \overrightarrow{C_6A_3}\right| \\
&\quad + \frac{2}{9}\left|\overrightarrow{C_7A_3} \quad \overrightarrow{C_5C_9}\right| + \frac{4}{9}\left|\overrightarrow{C_9C_6} \quad \overrightarrow{C_9C_7}\right| \\
Q_{23} &= \frac{1}{3}\left|\overrightarrow{A_4C_7} \quad \overrightarrow{C_6A_3}\right| + \frac{2}{3}\left|\overrightarrow{C_7A_3} \quad \overrightarrow{C_9C_7}\right| \\
Q_{30} &= \left|\overrightarrow{C_5A_2} \quad \overrightarrow{A_2C_6}\right| \\
Q_{31} &= \frac{1}{3}\left|\overrightarrow{C_5A_2} \quad \overrightarrow{C_6A_3}\right| + \frac{2}{3}\left|\overrightarrow{C_9C_6} \quad \overrightarrow{A_2C_6}\right| \\
Q_{32} &= \frac{1}{3}\left|\overrightarrow{C_7A_3} \quad \overrightarrow{A_2C_6}\right| + \frac{2}{3}\left|\overrightarrow{C_9C_6} \quad \overrightarrow{C_6A_3}\right| \\
Q_{33} &= \left|\overrightarrow{C_7A_3} \quad \overrightarrow{C_6A_3}\right|
\end{aligned}$$

It can be observed that the number of determinants in the coefficients is 36, i.e. 6×6 or $(4 \times 1 + 4 \times 2{\times}2 + 4 \times 4)$, one term for a corner, 2 terms for an edge and 4 terms for an internal coefficient, or again and in other words, all the combinations of all the vectors that can be constructed with the control points in all the directions.

A precise observation of the "corner" coefficients (such as Q_{00}) reveals that such coefficients give a control of the incident tangents thus detecting a potential intersection between the two incident (edge) curves. The other coefficients allow for a more precise control of the overall geometry.

[2] The true coefficients are $N_{IJ} = 4\,Q_{IJ}$.

From this Bézier formulation of the jacobian, Relation (18), we derive a sufficient condition of positivity.

Validity condition A 9-node quad element is valid if the four "corner" coefficients are strictly positive, while the others are non-negative. More precisely, the element is valid if Q_{00}, Q_{30}, Q_{33} and Q_{03} are strictly positive, while the other Q_{IJ} are non-negative, giving therefore 16 conditions (this is an immediate consequence of the convex hull property of Bézier surfaces).

Note that these conditions are not equivalent to having the jacobian positive at all nodes but are more demanding (while including this fact).

Refining the condition Since this condition is only a sufficient condition, it could be too restrictive. This is why, in some cases, it is possible to refine the condition so as to progressively approach a necessary and sufficient condition.

To make the refinement process clear, let us first consider the case where an edge coefficient is negative and more precisely the case of edge A_1A_2, i.e. $v = 0$. Along this edge, the jacobian polynomial reads (remember that $N_{ij} = 4Q_{ij}$)

$$\mathcal{J}(u,0) = (1-u)^3 N_{00} + 3u(1-u)^2 N_{10} + 3u^2(1-u)N_{20} + u^3 N_{30},$$

let us then assume that N_{10} is negative. The process is as follows:

– compute $\mathcal{J}(\frac{1}{2},0)$, if this value is not strictly positive, the element is not valid, STOP.
– if not, we cut the polynomial into two parts, one living in $[0,\frac{1}{2}]$, the other in $[\frac{1}{2},1]$ and we define the two sub-polynomials by computing their new control coefficients,
– having these coefficients, we observe their signs and iterate the process if necessary.

Hence, we have to compute the new coefficients, this is the matter of the classical De Casteljau refinement algorithm. In $[0,\frac{1}{2}]$, we define the sequence (evaluation at point $\frac{1}{2}$)

$$N^1_{00} = \frac{N_{00}+N_{10}}{2}, N^1_{10} = \frac{N_{10}+N_{20}}{2}, N^1_{20} = \frac{N_{20}+N_{30}}{2},$$
$$\text{then } N^2_{00} = \frac{N^1_{00}+N^1_{10}}{2}, N^2_{10} = \frac{N^1_{10}+N^1_{20}}{2}$$
$$\text{and, finally } N^3_{00} = \frac{N^2_{00}+N^2_{10}}{2}.$$

Note that N^3_{00} is exactly $\mathcal{J}(\frac{1}{2},0)$. The first part of the polynomial is replaced by

$$\mathcal{J}(u,0) = (1-u)^3 N_{00} + 3u(1-u)^2 N^1_{00} + 3u^2(1-u)N^2_{00} + u^3 N^3_{00},$$

and, similarly, for the second part, we have

$$\mathcal{J}(u,0) = (1-u)^3 N^3_{00} + 3u(1-u)^2 N^2_{10} + 3u^2(1-u)N^1_{20} + u^3 N_{30}.$$

It is trivial to check that these two polynomials are identical to the initial one, each in its ranging interval. Now, the important result is that the new coefficients are "closer" to the polynomial as can be seen by interpreting the polynomial as the Bézier curve $\gamma(u) = (u, \mathcal{J}(u,0))$. Therefore, we have a finer analysis of this curve or, in other words, of the sign of the polynomial.

The same idea applies when an internal coefficient is negative. We introduce a partition of the element into four parts and repeat the same process (while being more technically complex in this case).

An alternative method to compute the control coefficients As pointed out in [7] and [19], there is an alternative method to obtain the coefficients. Instead of a direct calculation (as before), we solve a linear system whose rank is that of the jacobian. Actually, as the degree is 3×3, the jacobian polynomial is made up of 16 terms. The idea is then to compute the jacobian of the Q2 quad at the nodes of a Q3 quad so as to have relations such as

$$\mathcal{J}(u_k, v_k) = \sum_{I=0,3}\sum_{J=0,3} B^3_I(u_k)B^3_J(v_k)N_{IJ}$$

where the couples (u_k, v_k) stand for the nodes of the Q3 quad, i.e. are all the points defined by the various combinations of the values $[0,\frac{1}{3},\frac{2}{3},1]$. In practice, however, the corner coefficients reduce to the jacobians themselves (as an example, $N_{00} = \mathcal{J}(0,0)$), while the edge coefficients result from a 2×2 system that is easy to solve by hand. As an example, for edge A_1A_2, we have $v_k = 0$ and we consider the two couples $(\frac{1}{3},0), (\frac{2}{3},0)$ leading to the system

$$\begin{cases} 27\mathcal{J}(\hat{B}_5) = 8N_{00} + 12N_{10} + 6N_{20} + N_{30} \\ 27\mathcal{J}(\hat{B}_6) = N_{00} + 6N_{10} + 12N_{20} + 8N_{30} \end{cases}$$

where $\hat{B}_5$ and $\hat{B}_6$ correspond to the above two couples. Then, we have

$$N_{10} = \frac{1}{6}(-5N_{00} + 18\mathcal{J}(\hat{B}_5) - 9\mathcal{J}(\hat{B}_6) + 2N_{30})$$
$$\text{and } N_{20} = \frac{1}{6}(2N_{00} - 9\mathcal{J}(\hat{B}_5) + 18\mathcal{J}(\hat{B}_6) - 5N_{30})$$

and similar expressions for the other edge coefficients. Once all of these have been computed, we have a 4×4 system to solve in order to obtain the four remaining coefficients.

5 Bidimensional complete Lagrange elements

The shape functions for Lagrange triangles together with for Lagrange quads can be written using a generic formulation. Let

$$\phi_i(u) = \frac{(-1)^i}{i!(d-i)!}\Pi_{l=0,l\neq i}^{l=d}(l-du) \text{ for } i \neq 0 \text{ and } \phi_0(u) = 1,$$

then, the shape function with index ij for a quad of degree d simply reads

$$p_{ij}(u,v) = \phi_i(u)\phi_j(v). \quad (20)$$

The generic expression for a triangle of degree d involves using the system of *barycentric* coordinates. Let

$$\phi_i(u) = \frac{1}{i!}\Pi_{l=0}^{i-1}(du-l),$$

then, the shape function with index ijk for a triangle of degree d simply reads

$$p_{ijk}(u,v,w) = \phi_i(u)\phi_j(v)\phi_k(w). \quad (21)$$

As pointed out in the above example, those expressions are not easy to consider if one wants to compute the derivatives and the jacobian polynomial, and therefore, we will write those elements in their Bézier forms.

5.1 Order 1 or 3-node triangle

The Bézier form and the finite element setting are the same in this case *when* the finite element form is written in terms of the barycentric coordinates. Indeed, we have $p_0(u,v,w) = u$, $p_1(u,v,w) = v$ and $p_2(u,v,w) = w$ which are exactly $B_{100}^1(u,v,w), B_{010}^1(u,v,w), B_{001}^1(u,v,w)$ and the 3 nodes are the 3 control points. Therefore, the 3-node triangle reads

$$\Sigma = \sigma(u,v,w) = \sum_{i+j+k=1} B_{ijk}^1(u,v,w)P_{ijk}, \quad (22)$$

where the P_{ijk}s are structured as follows:

P_{001}

$P_{100} \quad P_{010}.$

To become familiar with a system of barycentric coordinates, we fully detail this simple case. Let $(\hat{x},\hat{y})$ be the (cartesian) coordinates in $\hat{K}$, then the jacobian (in terms of F_K) is the following determinant:

$$\mathcal{J} = \mathcal{J}(\hat{x},\hat{y}) = \left| \frac{\partial F_K}{\partial \hat{x}} \; \frac{\partial F_K}{\partial \hat{y}} \right|,$$

and we consider (in terms of σ) the Jacobian matrix

$$[d\sigma] = \left[\frac{\partial \sigma}{\partial u} \; \frac{\partial \sigma}{\partial v} \; \frac{\partial \sigma}{\partial w} \right],$$

and we apply the variable change ($u = 1 - \hat{x} - \hat{y}$, $v = \hat{x}$, and $w = \hat{y}$), denoted by g, then

$$\mathcal{J} = [d\sigma]\,[dg]$$

holds where

$$[dg] = \begin{bmatrix} \frac{\partial u}{\partial \hat{x}} & \frac{\partial u}{\partial \hat{y}} \\ \frac{\partial v}{\partial \hat{x}} & \frac{\partial v}{\partial \hat{y}} \\ \frac{\partial w}{\partial \hat{x}} & \frac{\partial w}{\partial \hat{y}} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

therefore,

$$\mathcal{J} = \left| \frac{\partial \sigma}{\partial v} - \frac{\partial \sigma}{\partial u} \quad \frac{\partial \sigma}{\partial w} - \frac{\partial \sigma}{\partial u} \right|$$

hence, we have

$$\mathcal{J} = (-1)^2 \left| \frac{\partial \sigma}{\partial u} - \frac{\partial \sigma}{\partial w} \quad \frac{\partial \sigma}{\partial v} - \frac{\partial \sigma}{\partial w} \right|,$$

which is a simpler form to express the derivatives since σ and its partials are involved. Using the Relations (12), we obtain

$$\mathcal{J} = |P_{100} - P_{001} \quad P_{010} - P_{001}| = \left| \overrightarrow{P_{100}P_{010}} \; \overrightarrow{P_{100}P_{001}} \right|,$$

and as expected, the jacobian polynomial is constant and is twice the surface area of the element in hand. The validity condition resumes to $\mathcal{J} > 0$, remember that $P_{ijk} = A_{ijk}$.

5.2 Order d triangles

The finite element (as being complete) is written in a Bézier form as

$$\Sigma = \sigma(u,v,w) = \sum_{i+j+k=d} B_{ijk}^d(u,v,w)P_{ijk}, \quad (23)$$

where the P_{ijk}s are structured as shown in Sect. 2 where we also displayed the partials of such a function σ. We repeat what we did for the 3-node triangle to obtain the jacobian polynomial, we find

$$\mathcal{J}(u,v,w) = \sum_{I+J+K=2(d-1)} B_{IJK}^{2(d-1)}(u,v,w)N_{IJK},$$

where the coefficients N_{IJK} are

$$N_{IJK} = d^2 \sum_{i_1+i_2=I, j_1+j_2=J, k_1+k_2=K} \frac{C_{i_1j_1k_1}^{d-1} C_{i_2j_2k_2}^{d-1}}{C_{i_1+i_2,j_1+j_2,k_1+k_2}^{2(d-1)}} \left| \Delta_{i_1+1,j_1k_1}^{100} \quad \Delta_{i_2+1,j_2k_2}^{010} \right| \quad (24)$$

with $i_1 + j_1 + k_1 = i_2 + j_2 + k_2 = d - 1$ and with the following Δ:

Table 1 Statistics about the triangles of degree 1 to 6

d	#nodes	q	#coef	#terms
1	3	0	1	1
2	6	2	6	9
3	10	4	15	36
4	15	6	28	100
5	21	8	40	225
6	28	10	66	441

$$\Delta_{ijk}^{100} = \overrightarrow{P_{ijk}P_{i-1,j+1,k}} \quad \text{and} \quad \Delta_{ijk}^{010} = \overrightarrow{P_{ijk}P_{i-1,j,k+1}}.$$

The jacobian is a polynomial of degree $q = 2(d-1)$ and the number of control coefficients is $\frac{(q+1)\times(q+2)}{2}$ which rapidly increases with q or d as shown in the following Table 1 where we indicate also the number of terms (e.g. $(\frac{d\times(d+1)}{2})^2$) to be computed (some coefficients reduce to one term while others are the summation of a number of terms). The geometry of the observed element is valid if the "corner" coefficients are strictly positive, while the others are non-negative. As pointed out for the 9-node quad, the "corner" coefficients control the two tangents incident to a vertex and it is possible to refine (while being a rather technical step) the condition in the case where a non-"corner" coefficient is negative, the "corner" coefficients being all strictly positive and the jacobian evaluated at the cutting node(s) used in the refinement procedure being positive.

In practice Given the nodes of the element, one should be able to define the corresponding control points. This implies inverting the matrix defining the nodes in terms of the control points. Indeed, we have

$$\{A\} = [M]\{P\},$$

where $\{A\} = \{A_{ijk}\}_{ijk}$ and $\{P\} = \{P_{ijk}\}_{ijk}$. Since

$$\sigma(u,v,w) = \sum_{i+j+k=d} B_{ijk}^d(u,v,w)P_{ijk},$$

we have

$$A_{klm} = \sum_{i+j+k=d} B_{ijk}^d(u,v,w)P_{ijk}$$

for the triple $(u,v,w) = (\frac{k}{d},\frac{l}{d},\frac{m}{d})$,

thus giving the above matrix, and the solution is $\{P\} = [M]^{-1}\{A\}$. Actually, it is not strictly required to inverse the entire matrix, but instead to consider two systems of a lower dimension, one for the edge nodes, and, having the solution, the other for the internal nodes. For instance, consider $w = 0$ and the corresponding edge, then because some of the Bernstein are null, the system to be considered reduces to a $(d-1)\times(d-1)$ system. This system gives the expression of the (edge) nodes in terms of the (edge) control points, and the inverse gives the expression of the (edge) control points in terms of the (edge) nodes. Now, we consider the internal nodes (if any) and this result in a $(\frac{(d+1)(d+2)}{2} - 3d)\times(\frac{(d+1)(d+2)}{2} - 3d)$ system giving the expression of the (internal) nodes in terms of the (edge and internal) control points and the inverse gives the expression of the (internal) control points in terms of the (internal) nodes and the (edge) control points. Replacing those points by the solution of the first system results in the solution. The inverse matrices are computed once and applied to all the elements in the mesh or one would prefer using explicit formulae (see Appendix).

In the case of a straight-sided element (and provided the reference nodes are well located), the element is valid only if its 1-order associated element has a positive surface area and, therefore, the geometric validity is obtained for free (see hereafter how to minimize the cost of the validity control).

5.3 Order 1 or 4-node quadrilateral

The Bézier form and the finite element setting also coincide in this case. Indeed, we have $p_{00}(u,v) = (1-u)(1-v)$, $p_{10}(u,v) = u(1-v)$, $p_{11}(u,v) = uv$ and $p_{01}(u,v) = (1-u)v$ which are exactly $B_{00}^1(u)B_{00}^1(v),\ldots$ and the 4 nodes are the 4 control points. Therefore, the 4-node quad reads

$$\Sigma = \sigma(u,v) = \sum_{i=0,1}\sum_{j=0,1} B_i^1(u)B_j^1(v)P_{ij}, \tag{25}$$

where the P_{ijk}s are structured as follows:

$$\begin{matrix} P_{01} & P_{11} \\ P_{00} & P_{10}, \end{matrix}$$

and, the jacobian polynomial has the form[3]

$$\mathcal{J}(u,v) = \sum_{I=0,1}\sum_{J=0,1} B_I^1(u)B_J^1(v)N_{IJ},$$

and therefore, 4 terms and the control coefficients are

$$N_{IJ} = \sum_{i_1+i_2=I}\sum_{j_1+j_2=J} \begin{vmatrix}\Delta_{i_1,j_1}^{1,0} & \Delta_{i_2,j_2}^{0,1}\end{vmatrix}$$

for $I = 0,1$ and $J = 0,1$,

with $\Delta_{i,j}^{1,0} = \overrightarrow{P_{ij}P_{i+1,j}}$ and $\Delta_{i,j}^{0,1} = \overrightarrow{P_{ij}P_{i,j+1}}$,

therefore, those coefficients (after some permutations) are

$$N_{00} = |\overrightarrow{P_{00}P_{10}} \quad \overrightarrow{P_{00}P_{01}}|, N_{10} = |\overrightarrow{P_{00}P_{10}} \quad \overrightarrow{P_{10}P_{11}}|,$$

[3] note that this is exactly the same form as the element, this fact is true only for the degree 2.

$N_{11} = |\overrightarrow{P_{01}P_{10}} \quad \overrightarrow{P_{01}P_{11}}|, N_{01} = \left|\overrightarrow{P_{01}P_{00}} \quad \overrightarrow{P_{01}P_{11}}\right|.$

These coefficients measure twice the surface areas of the 4 triangles constructed with the vertices of the quad, and the validity condition resumes to have these 4 values positive, and therefore, the validity condition is, with no surprise, to have the element convex (with $P_{ij} = A_{ij}$ in the formula).

5.4 Order d quadrilaterals

The finite element[4] (as being complete) is written in a Bézier form as

$$\sigma(u, v) = \sum_{i=0,d} \sum_{j=0,d} B_i^d(u) B_j^d(v) P_{i,j}, \tag{26}$$

and the jacobian polynomial has the form

$$\mathcal{J}(u, v) = \sum_{I=0,q} \sum_{J=0,q} B_I^q(u) B_J^q(v) N_{IJ},$$

where $q = 2d - 1$ and the coefficient N_{IJ}s are

$$N_{IJ} = d^2 \sum_{i_1+i_2=I} \sum_{j_1+j_2=J} \frac{C_{i_1}^{d-1} C_{i_2}^d}{C_{i_1+i_2}^q} \frac{C_{j_1}^d C_{j_2}^{d-1}}{C_{j_1+j_2}^q}$$

$$\left|\Delta_{i_1,j_1}^{1,0} \quad \Delta_{i_2,j_2}^{0,1}\right| \text{ for } I = 0, q \text{ and } J = 0, q \tag{27}$$

with $\Delta_{i,j}^{1,0} = \overrightarrow{P_{ij}P_{i+1,j}}$ and $\Delta_{i,j}^{0,1} = \overrightarrow{P_{ij}P_{i,j+1}}$,

and with $i_1 = 0, d-1, i_2 = 0, d, j_1 = 0, d, j_2 = 0, d-1$. The jacobian is a polynomial of degree $q \times q = (2d-1) \times (2d-1)$, and the number of control coefficients is $(q+1)^2$ which rapidly increases with q or d as shown in the following Table 2 where we indicate also the number of terms (e.g. $d^2(d+1)^2$) to be computed (some coefficients reduce to one term, while others are the summation of a number of terms). The geometry of the observed element is valid if the "corner" coefficients are strictly positive, while the others are non-negative. As already pointed out, the "corner" coefficients control the two tangents incident to a vertex, and it is possible to refine (while being a rather technical step) the condition in the case where a non-"corner" coefficient is negative, the "corner" coefficients being all strictly positive.

Remark As compared (see the two Tables 1 , 2) with triangles, quadrilaterals have a jacobian with a higher degree and, in turn, the number of control coefficients is largely much more high for a given degree.

In practice Given the nodes of the element, one should be able to define the corresponding control points. This implies inverting the matrix defining the nodes in terms of the control points. Indeed, we have

$\{A\} = [M]\{P\},$

where $\{A\} = \{A_{ij}\}_{ij}$ and $\{P\} = \{P_{ij}\}_{ij}$. Since

$$\sigma(u, v) = \sum_{i=0,d} \sum_{j=0,d} B_i^d(u) B_j^d(v) P_{ij},$$

we have

$$A_{kl} = \sum_{i=0,d} \sum_{j=0,d} B_i^d(u) B_j^d(v) P_{ij}$$

for the couple $(u, v) = (\frac{k}{d}, \frac{l}{d})$, thus giving the above matrix and the solution is $\{P\} = [M]^{-1}\{A\}$.

Actually and as for triangles, it is not strictly required to inverse the entire matrix but, instead to consider two systems of a lower dimension, one for the edge nodes, and, having the solution, the other for the internal nodes. For instance, consider $w = 0$ and the corresponding edge, then because some of the Bernstein are null, the system to be considered reduces to a $(d-1) \times (d-1)$ system. This system gives the expression of the (edge) nodes in terms of the (edge) control points, and the inverse gives the expression of the (edge) control points in terms of the (edge) nodes. Now, we consider the internal nodes (if any) and this results in a $(d-1)^2 \times (d-1)^2$ system giving the expression of the (internal) nodes in terms of the (edge and internal) control points, and the inverse gives the expression of the (internal) control points in terms of the (internal) nodes and the (edge) control points. Replacing those points by the solution of the first system results in the solution. The inverse matrices are computed once and applied to all the elements in the mesh or one would prefer using explicit formulae (see Appendix).

In the case of a straight-sided element (and provided the reference nodes are well located), the element is valid only if its 1-order associated element is convex (therefore 4 surface areas must be computed) and, thus, the geometric validity is obtained for free.

Table 2 Statistics about the quadrilaterals of degree 1 to 6

d	#nodes	$q \times q$	#coef	#terms
1	4	1 × 1	4	4
2	9	3 × 3	16	36
3	16	5 × 5	36	144
4	25	7 × 7	64	400
5	36	9 × 9	100	900
6	49	11 × 11	144	1764

[4] While not being necessary, we consider the case where the degree is the same in both directions.

6 Bidimensional incomplete Lagrange elements

Incomplete or reduced elements have a reduced number of nodes (typically, the edge nodes are those of the complete elements, while the number of internal nodes is zero or smaller than that in the complete element). Low-degree elements are well documented in the literature, at least for quad geometries (8-node quad of degree 2) and for the 9-node triangle of degree 3. The polynomial space is also of a smaller dimension as compared with the complete space.

There are different methods to define reduced elements among which we have the Serendipity elements where space P is rich enough to achieve a good level of precision. One method specifies space P and, given an adequate number of nodes, constructs the shape functions by solving an adequate system satisfying the desired properties. Another method makes use of Taylor expansions in order to eliminate the internal nodes. Whatever the method, the shape functions have a generic expression (such as (20), (21) for the complete Lagrange elements). Let us consider the case of a tensor-product complete element and let p^c_{ij} be its shape functions, then we have

$$p_{ij}(u,v) = p^c_{ij}(u,v) + \sum_{kl} \alpha^{kl}_{ij} p^c_{kl}(u,v), \tag{28}$$

where indices ij correspond to the edge[5] nodes and indices kl are those related to the internal[6] nodes of the complete element and α^{kl}_{ij} is a coefficient (of repartition, how p^c_{kl} contributes to p_{ij}). For reduced simplices, we have a similar generic expression

$$p_{ijk}(u,v,w) = p^c_{ijk}(u,v,w) + \sum_{lmn} \alpha^{lmn}_{ijk} p^c_{lmn}(u,v,w). \tag{29}$$

The topic of this paper is not to give a detailed discussion of reduced elements[7] but, instead, given a reduced element, to find the conditions that give guarantee about its geometric validity. The main idea is, given such an element in a mesh, to return to a complete element equivalent to this reduced one and then to apply what we did previously for complete elements. It turns out that this requires to properly *invent* the "missing" nodes and the "missing" control points.

6.1 Order d Serendipity triangles

To have at least one internal node, we need to have $d = 3$, so we meet the complete triangle of degree 3, the well-known 10-node triangle, where we have only one internal node and the numbering of the nodes is as follows:

```
         003                       003
      102   012       ==>       102   012
   201   111   021           201         021
300   210   120   030     300   210   120   030
```

A Taylor expansion, based on the fact that the reduced polynomial space must contain the space $P^{d-1} = P^2$ (in terms of the variables x and y, the span of P^2 is made up of $1, x, y, xy, x^2$ and y^2), is used to express the edge values of a generic function (let q be this function) in terms of the internal value of this function, cf. [4], leads to a relation[8] like

$$12q(\hat{A}_{111}) + 2\sum_{ijk\in\mathcal{V}} q(\hat{A}_{ijk} - 3\sum_{ijk\in\mathcal{E}} q(\hat{A}_{ijk}) = 0, \tag{30}$$

where $\hat{A}_{111}$ is the point, in the reference element ($\hat{K}$), of coordinates $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $\mathcal{V}$ stands for the set of vertices in $\hat{K}$ and $\mathcal{E}$ stands for the set of edge nodes in $\hat{K}$. Since the $p_{ijk}(u,v,w)$s enjoy the same properties, for all the indices ijk, we have

$$12p_{ijk}(\hat{A}_{111}) + 2\sum_{lmn\in\mathcal{V}} p_{ijk}(\hat{S}_{lmn}) - 3\sum_{lmn\in\mathcal{E}} p_{ijk}(\hat{A}_{lmn}) = 0.$$

Now, we use Relation (29) to replace the p_{ijk} by their counterpart in terms of the complete shape functions. For symmetry reasons, this relation reduces to

$$p_{ijk}(u,v,w) = p^c_{ijk}(u,v,w) + \alpha p^c_{111}(u,v,w) \quad \text{for} \quad ijk \in \mathcal{V},$$

and

$$p_{ijk}(u,v,w) = p^c_{ijk}(u,v,w) + \beta p^c_{111}(u,v,w) \quad \text{for} \quad ijk \in \mathcal{E},$$

in other words, there are only two coefficients. Let us fix $ijk = 300$, then, we have

$$12p_{300}(\hat{A}_{111}) + 2\sum_{lmn\in\mathcal{V}} p_{300}(\hat{S}_{lmn}) - 3\sum_{lmn\in\mathcal{E}} p_{300}(\hat{A}_{lmn}) = 0,$$

and this resumes to

$$\alpha = -\frac{1}{6},$$

and the same (fix $ijk = 210$) implies that $\beta = \frac{1}{4}$. With these values, we have the reduced shape functions fully defined via (29).

Then the reduced element seen as a patch reads

$$M(u,v,w) = \sum_{ijk} p_{ijk}(u,v,w) A_{ijk},$$

[5] actually, for some reduced elements, one or several internal nodes of the complete element are retained as nodes for the reduced element.

[6] cf. infra.

[7] and the way in which they are constructed, a paper being currently under preparation to do this

[8] the so-called Serendipity relation.

where ijk lives in $\mathcal{V}$ and $\mathcal{E}$, i.e. 9 indices. We replace again the p_{ijk} by means of the p^c_{ijk}, then we have

$$M(u,v,w) = \sum_{ijk} \Big(p^c_{ijk}(u,v,w) + \alpha_{ijk} p^c_{111}(u,v,w) \Big) A_{ijk},$$

with $\alpha_{ijk} = \alpha$ or β, then this reads also

$$M(.,.,.) = \sum_{ijk} p^c_{ijk}(u,v,w) A_{ijk} + \sum_{ijk} \alpha_{ijk} A_{ijk} p^c_{111}(u,v,w),$$

therefore, let

$$A_{111} = \sum_{ijk} \alpha_{ijk} A_{ijk},$$

so that

$$M(u,v,w) = \sum_{ijk} p^c_{ijk}(u,v,w) A_{ijk},$$

with now 10 indices. In other words, we have *invented* the node A_{111} with which we can define a complete element fully equivalent to the reduced element. As already seen, a complete element is equivalent to the Bézier patch

$$\sum_{i+j+k=3} B^3_{ijk}(u,v,w) P_{ijk},$$

from which we obtain P_{111} and it turns out that P_{111} simply reads as

$$P_{111} = \sum_{ijk} \alpha_{ijk} P_{ijk},$$

e.g. the same expression as A_{111}.

In practice The 9-node triangle is easy to analyse. Given its nodes, we find its edge control points by the formulae (see Appendix) and then we compute P_{111} using the above formula. Then, just use Relation (24) to have the control coefficients of the jacobian.

It is a rather technical task to discuss about higher order reduced triangles, and therefore, we do not pursuit this story of reduced triangles, see [17].

6.2 Order d Serendipity quadrilaterals

For $d = 2$, we have the 9-node quad with one internal node and the numbering of the nodes is as follows:

```
02   12   22            02   12   22
01   11   21    ==>     01        21
00   10   20            00   10   20
```

As for the above triangle, in this case, we impose the space $P^d = P^2$ to be included in the reduced polynomial space. Then, the Taylor expansion gives the Serendipity relation

$$4q(\hat{A}_{11}) + \sum_{j=1,4} q(\hat{S}_j) - 2\sum_{j=1,4} q(\hat{A}_j) = 0, \tag{31}$$

where $\hat{A}_{11}$ is the point, in the reference element ($\hat{K}$), of coordinates $(\frac{1}{2}, \frac{1}{2})$, $\hat{S}_j$ is vertex j of the reference element and the $\hat{A}_j$s are the edge nodes of the reference element. Since the $p_{ij}(u,v)$s enjoy the same properties, for all the indices ij, we have

$$4p_{ij}(\hat{A}_{11}) + \sum_{lm \in \mathcal{V}} p_{ij}(\hat{S}_{lm}) - 2\sum_{lm \in \mathcal{E}} p_{ij}(\hat{A}_{lm}) = 0,$$

where $\mathcal{V}$ stands for the set of vertices in $\hat{K}$ and $\mathcal{E}$ stands for the set of edge nodes in $\hat{K}$. Now, we use Relation (28) to replace the p_{ij} by their counterpart in terms of the complete shape functions. For symmetry reasons, this relation reduces to

$$p_{ij}(u,v) = p^c_{ij}(u,v) + \alpha\, p^c_{11}(u,v) \quad \text{for} \quad ij \in \mathcal{V},$$
$$\text{and} \quad p_{ij}(u,v) = p^c_{ij}(u,v) + \beta\, p^c_{11}(u,v) \quad \text{for} \quad ij \in \mathcal{E},$$

in other words, there are only two coefficients. Let us fix ij = 00, then, we have

$$4p_{00}(\hat{A}_{11}) + \sum_{lm \in \mathcal{V}} p_{00}(\hat{S}_{lmn}) - 2\sum_{lm \in \mathcal{E}} p_{00}(\hat{A}_{lm}) = 0,$$

and this resumes to

$$\alpha = -\frac{1}{4},$$

and the same (fix $ij = 10$) implies that $\beta = \frac{1}{2}$. With these values, we have the reduced shape functions fully defined via (28).

Then, the reduced element, seen as a patch reads

$$M(u,v) = \sum_{ij} p_{ij}(u,v) A_{ij},$$

where ij lives in $\mathcal{V}$ and $\mathcal{E}$, i.e. 8 indices. We replace again the p_{ij} by means of the p^c_{ij}, then we have

$$M(u,v) = \sum_{ij} (\, p^c_{ij}(u,v) + \alpha_{ij} p^c_{11}(u,v)\,) A_{ij},$$

with $\alpha_{ij} = \alpha$ or β, then this reads also

$$M(u,v) = \sum_{ij} p^c_{ij}(u,v) A_{ij} + \sum_{ij} \alpha_{ij} A_{ij} p^c_{11}(u,v),$$

therefore, let

$$A_{11} = \sum_{ij} \alpha_{ij} A_{ij},$$

so that

$$M(u,v) = \sum_{ij} p^c_{ij}(u,v) A_{ij},$$

with now 9 indices. In other words, we have *invented* the node A_{11} with which we can define a complete element fully equivalent to the reduced element. As we already seen, a complete element is equivalent to the Bézier patch

$$\sum_{i=0,2}\sum_{j=0,2} B_i^2(u)B_j^2(v)P_{ij},$$

from which we obtain P_{11} and it turns out that P_{11} simply reads as

$$P_{11} = \sum_{ij} \alpha_{ij} P_{ij},$$

e.g. the same expression as A_{11}.

In practice The 8-node quad is easy to analyse. Given its nodes, we find its edge control points by the formulae (see Appendix) and then we compute P_{11} using the above formula. Then, just use Relation (27) to have the control coefficients of the jacobian.

Higher order reduced quadrilaterals can be defined but, as it is for triangles, this is a rather technical task, and therefore, we do not pursuit this story of reduced quads, see [1] or [13] for the Serendipity family.

7 Tridimensional complete Lagrange elements

Formulae (20) and (21) extend to hexahedra and simplices and give the shape functions.

7.1 Order 1 or 4-node tetrahedron

We play again the story of the 3-node triangle. The Bézier form and the finite element setting coincide in this case *when* the finite element form is written in terms of the barycentric coordinates. Indeed, we have $p_0(u,v,w,t) = u$, $p_1(u,v,w,t) = v$, $p_2(u,v,w,t) = w$ and $p_3(u,v,w,t) = t$ which are exactly $B^1_{1000}(u,v,w)$, $B^1_{0100}(u,v,w)$, $B^1_{0010}(u,v,w)$ and $B^1_{0001}(u,v,w)$ and the 4 nodes are the 4 control points. Therefore, the 4-node tet reads

$$\Theta = \theta(u,v,w,t) = \sum_{i+j+k+l=1} B^1_{ijkl}(u,v,w,t)P_{ijkl}. \tag{32}$$

Let $(\hat{x},\hat{y},\hat{z})$ be the (cartesian) coordinates in $\hat{K}$, then the jacobian (in terms of F_K) is the following determinant:

$$\mathcal{J} = \mathcal{J}(\hat{x},\hat{y},\hat{z}) = \left| \frac{\partial F_K}{\partial \hat{x}} \;\; \frac{\partial F_K}{\partial \hat{y}} \;\; \frac{\partial F_K}{\partial \hat{z}} \right|,$$

while (in terms of θ) we consider the matrix

$$[d\theta] = \left[\frac{\partial \theta}{\partial u} \;\; \frac{\partial \theta}{\partial v} \;\; \frac{\partial \theta}{\partial w} \;\; \frac{\partial \theta}{\partial t} \right],$$

and with the variable change ($u = 1-\hat{x}-\hat{y}-\hat{z}$, $v=\hat{x}$, $w=\hat{y}$, and $t=\hat{z}$), denoted by g, we have

$$\mathcal{J} = [d\theta]\,[dg],$$

where

$$[dg] = \begin{bmatrix} \frac{\partial u}{\partial \hat{x}} & \frac{\partial u}{\partial \hat{y}} & \frac{\partial u}{\partial \hat{z}} \\ \frac{\partial v}{\partial \hat{x}} & \frac{\partial v}{\partial \hat{y}} & \frac{\partial v}{\partial \hat{z}} \\ \frac{\partial w}{\partial \hat{x}} & \frac{\partial w}{\partial \hat{y}} & \frac{\partial w}{\partial \hat{z}} \\ \frac{\partial t}{\partial \hat{x}} & \frac{\partial t}{\partial \hat{y}} & \frac{\partial t}{\partial \hat{z}} \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

therefore

$$\mathcal{J} = \left| \frac{\partial \theta}{\partial v} - \frac{\partial \theta}{\partial u} \;\;\; \frac{\partial \theta}{\partial w} - \frac{\partial \theta}{\partial u} \;\;\; \frac{\partial \theta}{\partial t} - \frac{\partial \theta}{\partial u} \right|$$

and after manipulating this determinant, we find

$$\mathcal{J} = (-1)^3 \left| \frac{\partial \theta}{\partial u} - \frac{\partial \theta}{\partial t} \;\;\; \frac{\partial \theta}{\partial v} - \frac{\partial \theta}{\partial t} \;\;\; \frac{\partial \theta}{\partial w} - \frac{\partial \theta}{\partial t} \right|,$$

which is a simpler form to express the derivatives since θ and its partials are involved. Using relations (now in three dimensions) like the Relations (12), we obtain

$$\begin{aligned} \mathcal{J} &= -\left| P_{1000} - P_{0001} \quad P_{0100} - P_{0001} \quad P_{0010} - P_{0001} \right| \\ &= \left| \overrightarrow{P_{1000}P_{0100}} \quad \overrightarrow{P_{1000}P_{0010}} \quad \overrightarrow{P_{1000}P_{0001}} \right|, \end{aligned}$$

as expected, the jacobian polynomial is constant and is (at a scaling factor) the volume of the element in hand. The validity condition resumes to $\mathcal{J} > 0$ with $P_{ijkl} = A_{ijkl}$ in the formula.

7.2 Order 1 or 6-node prism

The Bézier form of a 6-node prism or pentahedron involves a barycentric form in $\hat{x},\hat{y}$ and a tensor-product form in $\hat{z}$, and therefore, with adequate notations, we have

$$\Theta(u,v,w,t) = \sum_{i+j+k=1}\sum_{l=0,1} B^1_{ijk}(u,v,w)B^1_l(t)P_{ijkl}, \tag{33}$$

with $u+v+w=1, u=1-\hat{x}-\hat{y}, v=\hat{x}, w=\hat{y}$ and $t=\hat{z}$, and P_{ijkl} the control points (i.e. the vertices). The jacobian reads

$$\mathcal{J} = (-1)^2 \left| \frac{\partial \theta}{\partial u} - \frac{\partial \theta}{\partial w} \;\;\; \frac{\partial \theta}{\partial v} - \frac{\partial \theta}{\partial w} \;\;\; \frac{\partial \theta}{\partial t} \right|,$$

and using the formulae in Sect. 2, we have

$$\mathcal{J} = \Bigg| \sum_{l_1=0,1} B^1_{l_1}(t)(P_{100l_1} - P_{001l_1}).$$
$$\sum_{l_2=0,1} B^1_{l_2}(t)(P_{010l_2} - P_{001l_2})$$
$$\sum_{i+j+k=1} B^1_{ijk}(u,v,w)B^0_0(t)\overrightarrow{P_{ijk0}P_{ijk1}} \Bigg|,$$

or

$$\mathcal{J} = \sum_{i+j+k=1} \sum_{l_1=0,1} \sum_{l_2=0,1} B^1_{l_1}(t)B^1_{l_2}(t)B^1_{ijk}(u,v,w)$$
$$\left| \overrightarrow{P_{100l_1}P_{001l_1}} \quad \overrightarrow{P_{010l_2}P_{001l_2}} \quad \overrightarrow{P_{ijk0}P_{ijk1}} \right|,$$

or again

$$\mathcal{J}(u,v,w,t) = \sum_{I+J+K=1} \sum_{L=0,2} B^1_{IJK}(u,v,w)B^2_L(t)N_{IJKL}$$

where $I=i, J=j, K=k$ and $L=l_1+l_2$ and with the control coefficients

$$N_{IJKL} = \sum_{l_1+l_2=L} \frac{C^1_{l_1}C^1_{l_2}}{C^2_{l_1+l_2}} \left| \overrightarrow{P_{100l_1}P_{001l_1}} \; \overrightarrow{P_{010l_2}P_{001l_2}} \; \overrightarrow{P_{IJK0}P_{IJK1}} \right|$$

which also reads

$$N_{IJKL} = \sum_{l_1+l_2=L} \frac{1}{C^2_L} \left| \overrightarrow{P_{100l_1}P_{010l_1}} \; \overrightarrow{P_{100l_2}P_{001l_2}} \; \overrightarrow{P_{IJK0}P_{IJK1}} \right|.$$

The degree of the jacobian polynomial is 1 in (u, v, w) and 2 in t, the number of control coefficients is 9. The element is valid if its 6 corner coefficients are positive, while the 3 others are non-negative. Note that 12 determinants[9] have to be computed to obtain the 9 control coefficients. A corner coefficient, like N_{0000}, measures the volume of tetrahedron $P_{0000}P_{0100}P_{0010}P_{0001}$ (at a scaling factor). The "vertical" edges have a control coefficient with 2 terms, for example, we have

$$N_{1001} = \frac{1}{2} |\overrightarrow{P_{1001}P_{0101}} \quad \overrightarrow{P_{1000}P_{0010}} \quad \overrightarrow{P_{1000}P_{1001}}|$$
$$+ \frac{1}{2} |\overrightarrow{P_{1000}P_{0100}} \quad \overrightarrow{P_{1001}P_{0011}} \quad \overrightarrow{P_{1000}P_{1001}}|.$$

Such a coefficient gives a control about a possible rotation (torsion) from one triangular face to the other or, in other words, the geometry of the quadrilateral faces. In the case, a quadrilateral face is planar, it is easy to see that the corresponding (e.g. the 3) edge coefficients are linear combinations of the corner coefficients, and therefore, only 6 coefficients are necessary to control the validity an element with planar faces. To prove this, just write, for example, in coefficient N_{1001}, $\overrightarrow{P_{1001}P_{0101}} = \alpha\overrightarrow{P_{1000}P_{0100}} + \beta\overrightarrow{P_{1000}P_{1001}}$ and replace $\overrightarrow{P_{1001}P_{0101}}$ by this expression in N_{1001}.

[9] This number of terms is exactly the number of combinations between the triples of all the vectors that can be constructed with the vertices of the element, e.g. 4 with respect to (u, v, w) and 3 with respect to t, therefore 4×3. Note that this property holds for all the elements and whatever the degree.

7.3 Order 1 or 8-node hexahedron

As will be shown (and, somehow, surprisingly), the geometric validity of this *a priori* simple element involves a large number of calculations. The Bézier form of a 8-node hexahedron is

$$\Theta(u,v,w) = \sum_{i=0,1} \sum_{j=0,1} \sum_{k=0,1} B^1_i(u)B^1_j(v)B^1_k(w)P_{ijk} \tag{34}$$

with u, v and w in [0, 1] and P_{ijk} the control points (i.e. the vertices). The jacobian reads

$$\mathcal{J}(u,v,w) = \left| \frac{\partial\theta}{\partial u} \quad \frac{\partial\theta}{\partial v} \quad \frac{\partial\theta}{\partial w} \right|,$$

and using the formulae in Sect. 2, we have

$$\frac{\partial\theta}{\partial u} = \sum_{j=0,1} \sum_{k=0,1} B^1_j(v)B^1_k(w)\overrightarrow{P_{0jk}P_{1jk}},$$

and then

$$\mathcal{J}(u,v,w) = \Bigg| \sum_{j=0,1} \sum_{k=0,1} B^1_j(v)B^1_k(w)\overrightarrow{P_{0jk}P_{1jk}}$$
$$\sum_{i=0,1} \sum_{k=0,1} B^1_i(u))B^1_k(w)\overrightarrow{P_{i0k}P_{i1k}}$$
$$\sum_{i=0,1} \sum_{j=0,1} B^1_i(u)B^1_j(v)\overrightarrow{P_{ij0}P_{ij1}} \Bigg|,$$

so

$$\mathcal{J}(u,v,w) = \sum_{j_1=0,1} \sum_{k_1=0,1} \sum_{i_2=0,1} \sum_{k_2=0,1} \sum_{i_3=0,1} \sum_{j_3=0,1}$$
$$B^1_{j_1}(v)B^1_{k_1}(w)B^1_{i_2}(u))B^1_{k_2}(w)B^1_{i_3}(u)B^1_{j_3}(v)$$
$$\left| \overrightarrow{P_{0j_1k_1}P_{1j_1k_1}} \quad \overrightarrow{P_{i_20k_2}P_{i_21k_2}} \quad \overrightarrow{P_{i_3j_30}P_{i_3j_31}} \right|,$$

and, by the multiplication rule, we have

$$\mathcal{J}(u,v,w) = \sum_{I=0,2} \sum_{J=0,2} \sum_{K=0,2} B^2_I(u)B^2_J(v)B^2_K(w)N_{IJK}$$

where the control coefficients are

$$N_{IJK} = \sum_{i_2+i_3=I} \sum_{j_1+j_3=J} \sum_{k_1+k_2=K} \frac{1}{C^2_I C^2_J C^2_K}$$
$$\left| \overrightarrow{P_{0j_1k_1}P_{1j_1k_1}} \quad \overrightarrow{P_{i_20k_2}P_{i_21k_2}} \quad \overrightarrow{P_{i_3j_30}P_{i_3j_31}} \right|.$$

The jacobian is a polynomial of degree 2 in each direction. There are 27 control coefficients involving 64 determinants ($4 \times 4 \times 4$). We have four type of coefficients, the corner, the edge and the face coefficients and one volume coefficient (N_{111}). The corner and the edge coefficients provide the same control as in the prism but, here, we have one coefficient for each face and one internal coefficient. In the case where the faces are planar, the edge and the face coefficients are redundant with the corner coefficients (same proof as for the prism) but we do not find any relation between N_{111} and those coefficients. It means that an element with planar faces is controlled by 9 coefficients (and not only 8).

7.4 Order 1 or 5-node pyramid

We find in various papers a definition for this 5-node element using rational polynomial as shape functions which leads to a singularity in the jacobian polynomial (which is ∞ for $w = 1$). Other references propose using classical Lagrange polynomials. This is our choice and we define the pyramid as an hexahedron with a degenerescence for $w = 1$. This way of construction leads to very simple calculations, while we still have a singularity (the jacobian is zero for $w = 1$).

Therefore, we use (34) with $P_{ij1} = P_{001}, \forall(i,j)$, then

$$\Theta(u,v,w)\Big| = \sum_{i=0,1}\sum_{j=0,1}\sum_{k=0,1} B_i^1(u)B_j^1(v)B_k^1(w)P_{ijk},$$

$$\Theta(u,v,w) = \sum_{i=0,1}\sum_{j=0,1} B_i^1(u)B_j^1(v)B_0^1(w)P_{ij0} + \sum_{i=0,1}\sum_{j=0,1} B_i^1(u)B_j^1(v)B_1^1(w)P_{ij1},$$

and $\sum_{i=0,1}\sum_{j=0,1} B_i^1(u)B_j^1(v)$ factorizes, so we simply have

$$\Theta(u,v,w) = \sum_{i=0,1}\sum_{j=0,1} B_i^1(u)B_j^1(v)B_0^1(w)P_{ij0} + B_1^1(w)P_{001} \tag{35}$$

from which we easily define the 5 shape functions.

The jacobian polynomial is that of the 8-node hexahedron but a number of terms are null so we have

$$\mathcal{J}(u,v,w) = \left| \sum_{j=0,1} B_j^1(v)B_0^1(w)\overrightarrow{P_{0j0}P_{1j0}} \quad \sum_{i=0,1} B_i^1(u))B_0^1(w)\overrightarrow{P_{i00}P_{i10}} \quad \sum_{i=0,1}\sum_{j=0,1} B_i^1(u)B_j^1(v)\overrightarrow{P_{ij0}P_{ij1}} \right|,$$

and then

$$\mathcal{J}(u,v,w) = \sum_{j_1=0,1}\sum_{i_2=0,1}\sum_{i_3=0,1}\sum_{j_3=0,1} B_{j_1}^1(v)B_0^1(w)B_{i_2}^1(u))B_0^1(w)B_{i_3}^1(u)B_{j_3}^1(v) \left|\overrightarrow{P_{0j_10}P_{1j_10}} \quad \overrightarrow{P_{i_200}P_{i_210}} \quad \overrightarrow{P_{i_3j_30}P_{i_3j_31}}\right|,$$

written as

$$\mathcal{J}(u,v,w) = \sum_{I=0,2}\sum_{J=0,2} B_I^2(u)B_J^2(v)B_0^2(w)N_{IJ0}$$

where the control coefficients are

$$N_{IJ0} = \sum_{i_2+i_3=I}\sum_{j_1+j_3=J} \frac{1}{C_I^2 C_J^2} \left|\overrightarrow{P_{0j_10}P_{1j_10}} \quad \overrightarrow{P_{i_200}P_{i_210}} \quad \overrightarrow{P_{i_3j_30}P_{i_3j_31}}\right|$$

The jacobian is a polynomial of degree 2 in each direction. *A priori*, there are 9 control coefficients involving 16 determinants ($2 \times 2 \times 4$). We have three types of coefficients, the corner and the (horizontal) edge coefficients and one face coefficient (N_{110}). The corner and the edge coefficients provide the same control as in the previous elements and we have one coefficient for the quadrilateral face. Actually, the edge coefficients are redundant with the (first four) corner coefficients, and the number of effective coefficients reduces to 5. The reason is due to the fact that all the "vertical" vectors meet at the "top" vertex (so that the "vertical" quadrilateral faces degenerate in triangular faces). Due to the polynomial $B_0^2(w) = (1 - w)^2$, the jacobian vanishes when $w = 1$.

7.5 Order d tetrahedra

We use the definition (23) to discuss the case of a degree d tetrahedron and, now, its reading is

$$\Theta = \theta(u,v,w,t) = \sum_{i+j+k+l=d} B_{ijkl}^d(u,v,w,t)P_{ijkl}, \tag{36}$$

where the P_{ijkl}s are the control points. We repeat what we did for the triangle of degree d to obtain the jacobian polynomial, and we find

$$\mathcal{J}(u,v,w,t) = \sum_{I+J+K+L=3(d-1)} B_{IJKL}^{3(d-1)}(u,v,w,t)N_{IJKL},$$

where the coefficients N_{IJKL} are

$$N_{IJKL} = d^3 \sum_{|i|=I,|j|=J,|k|=K,|l|=L} \frac{C_{i_1j_1k_1l_1}^{d-1} C_{i_2j_2k_2l_2}^{d-1} C_{i_3j_3k_3l_3}^{d-1}}{C_{IJKL}^{3(d-1)}} \left|\Delta_{i_1+1,j_1k_1l_1}^{1000} \quad \Delta_{i_2+1,j_2k_2l_2}^{0100} \quad \Delta_{i_3+1,j_3k_3l_3}^{0010}\right|, \tag{37}$$

with $|i| = i_1 + i_2 + i_3$, ..., and with $i_1 + j_1 + k_1 + l_1 = i_2 + j_2 + k_2 + l_2 = i_3 + j_3 + k_3 + l_3 = d - 1$ and with the following Δ:

$$\Delta^{1000}_{ijkl} = \overrightarrow{P_{ijkl}P_{i-1,j+1,k,l}}, \quad \Delta^{0100}_{ijkl} = \overrightarrow{P_{ijkl}P_{i-1,j,k+1,l}} \quad \text{and} \quad \Delta^{0010}_{ijkl} = \overrightarrow{P_{ijkl}P_{i-1,j,k,l+1}}.$$

The jacobian is a polynomial of degree $q = 3(d-1)$. The number of control coefficients is $\frac{(q+1)(q+2)(q+3)}{6}$, and the number of determinants involved in these coefficients is $(\frac{d\times(d+1)}{2} + \frac{(d-1)d}{2} + \cdots)^3$, see Table 3.

In practice Given the nodes of the element, one should be able to define the control points. For the control points corresponding to the edges, the formulae are identical to what we have in two dimensions (see Appendix), for those corresponding to a face, the solutions given for a triangle apply. For the eventual internal control points, one has to solve the corresponding system.

7.6 Order d prisms

The Bézier form of a degree d prism or pentahedron involves a barycentric form in $\hat{x}, \hat{y}$ and a natural form in $\hat{z}$, and therefore, with adequate notations (and the same degree in all directions), we have

$$\Theta(u,v,w,t) = \sum_{i+j+k=d} \sum_{l=0,d} B^d_{ijk}(u,v,w) B^d_l(t) P_{ijkl}, \tag{38}$$

with $u + v + w = 1, u = 1 - \hat{x} - \hat{y}, v = \hat{x}, w = \hat{y}$ and $t = \hat{z}$, and P_{ijkl} the control points (i.e. the vertices). The jacobian reads

$$\mathcal{J} = (-1)^2 \left| \frac{\partial\theta}{\partial u} - \frac{\partial\theta}{\partial w} \quad \frac{\partial\theta}{\partial v} - \frac{\partial\theta}{\partial w} \quad \frac{\partial\theta}{\partial t} \right|,$$

and using the formulae in Sect. 2, we have

$$\mathcal{J} = \left| \sum_{i_1+j_1+k_1=d-1} \sum_{l_1=0}^{d} B^{d-1}_{i_1 j_1,k_1}(u,v,w) B^d_{l_1}(t) \Delta^{100}_{i_1+1,j_1,k_1} \quad \sum_{i_2+j_2+k_2=d-1} \sum_{l_2=0}^{d} B^{d-1}_{i_2 j_2,k_2}(u,v,w) B^d_{l_2}(t) \Delta^{010}_{i_2+1,j_2,k_2} \quad \sum_{i_3+j_3+k_3=d} \sum_{l_3=0}^{d-1} B^d_{i_3 j_3,k_3}(u,v,w) B^{d-1}_{l_3}(t) \overrightarrow{P_{i_3 j_3 k_3 l_3} P_{i_3 j_3 k_3, l_3+1}} \right|$$

with

$$\Delta^{100}_{ijkl} = \overrightarrow{P_{ijkl}P_{i-1,j+1,k,l}} \text{ and } \Delta^{010}_{ijkl} = \overrightarrow{P_{ijkl}P_{i-1,j,k+1,l}}.$$

Grouping together the Bernstein leads to

$$\mathcal{J}(\ldots.) = \sum_{I+J+K=3d-2} \sum_{L=0}^{3d-1} B^{3d-2}_{IJK}(u,v,w) B^{3d-1}_L(t) N_{IJKL}$$

where the coefficients N_{IJKL} are

$$N_{IJKL} = d^3 \sum_{|i|=I,|j|=J,|k|=K} \sum_{|l|=L} \frac{C^{d-1}_{i_1 j_1 k_1} C^{d-1}_{i_2 j_2 k_2} C^d_{i_3 j_3 k_3} C^d_{l_1} C^d_{l_2} C^{d-1}_{l_3}}{C^{3d-2}_{IJK} C^{3d-1}_L}$$

$$\left| \Delta^{100}_{i_1+1,j_1 k_1 l_1} \quad \Delta^{010}_{i_2+1,j_2 k_2 l_2} \quad \Delta^{0001}_{i_3,j_3 k_3 l_3} \right|, \tag{39}$$

where $|i| = i_1 + i_2 + i_3$, ..., and with $i_1 + j_1 + k_1 = i_2 + j_2 + k_2 = i_3 + j_3 + k_3 = d - 1, l_1 = 0, d, l_2 = 0, d, l_3 = 0, d-1$ and with the following Δ:

$$\Delta^{100}_{ijkl} = \overrightarrow{P_{ijkl}P_{i-1,j+1,k,l}}, \quad \Delta^{010}_{ijkl} = \overrightarrow{P_{ijkl}P_{i-1,j,k+1,l}}$$

$$\text{and} \quad \Delta^{0001}_{ijkl} = \overrightarrow{P_{ijkl}P_{i,j,k,l+1}}.$$

The jacobian is a polynomial of degree $q \times q' = (3d-2) \times (3d-1)$ and the number of control coefficients is $(q'+1)\frac{(q+1)(q+2)}{2}$, while the number of determinants (component of the coefficients) is $\frac{1}{8}d^3(d+1)^5(d+2)$, see the table (Table 4).

Remark As compared (see Tables 3, 4) with tetrahedra, prisms have a jacobian with a higher degree and, in turn, the number of control coefficients is largely much more high for a given degree.

In practice Given the nodes of the element, one should be able to define the control points. As for the previous elements, the control points corresponding to the edges are obtained using the formulae we have in two dimensions (see Appendix) and for those corresponding to a face, the solutions given for triangular or a quadrilateral apply. For the eventual internal control points, one has to solve the corresponding system.

Table 3 Statistics about the tetrahedra of degree 1 to 5

d	#nodes	q	#coef	#terms
1	4	0	1	1
2	10	3	20	64
3	20	6	84	1,000
4	35	9	220	8,000
5	56	12	455	42,875

Table 4 Statistics about the pentahedra of degree 1 to 5

d	#nodes	$q \times q'$	#coef	#terms
1	6	1×2	9	12
2	18	4×5	90	972
3	40	7×8	324	17,280
4	75	10×11	792	150,000
5	126	13×14	1575	850,500

Table 5 Statistics about the hexahedra of degree 1 to 5

d	#nodes	$q \times q \times q$	#coef	#termss
1	8	$2 \times 2 \times 2$	27	64
2	27	$5 \times 5 \times 5$	216	5,832
3	64	$8 \times 8 \times 8$	512	110,592
4	125	$11 \times 11 \times 11$	1,331	1,000,000
5	216	$14 \times 14 \times 14$	2,744	5,832,000

7.7 Order d hexahedra

We use the definition (26) to discuss the case of a degree d hexahedron (with the same degree in the three directions), so we have

$$\theta(u,v,w) = \sum_{i=0,d}\sum_{j=0,d}\sum_{k=0,d} B_i^d(u)B_j^d(v)B_k^d(w)\,P_{ijk}, \tag{40}$$

and, the jacobian polynomial has the form

$$\mathcal{J}(u,v,w) = \sum_{I=0,q}\sum_{J=0,q}\sum_{K=0,q} B_I^q(u)B_J^q(v)B_K^q(w)N_{IJK},$$

where $q = 3d-1$ and the coefficients N_{IJK} are

$$N_{IJK} = d^3 \sum_{|i|=I}\sum_{|j|=J}\sum_{|k|=K} \frac{C_{i_1}^{d-1}C_{i_2}^{d}C_{i_3}^{d}}{C_I^q}\frac{C_{j_1}^{d}C_{j_2}^{d-1}C_{j_3}^{d}}{C_J^q}$$
$$\frac{C_{k_1}^{d}C_{k_2}^{d}C_{k_3}^{d-1}}{C_K^q}\left|\Delta_{i_1j_1k_1}^{100} \quad \Delta_{i_2j_2k_2}^{010} \quad \Delta_{i_3j_3k_3}^{001}\right|, \tag{41}$$
$$\text{with } \Delta_{ijk}^{100} = \overrightarrow{P_{ijk}P_{i+1,jk}}, \Delta_{ijk}^{010} = \overrightarrow{P_{ijk}P_{i,j+1,k}}$$
$$\text{and } \Delta_{ijk}^{001} = \overrightarrow{P_{ijk}P_{ij,k+1}},$$

$i_1 = 0, d-1, i_2 = 0, d, i_3 = 0, d, \ldots$. The jacobian is a polynomial of degree $q \times q \times q = (3d-1) \times (3d-1) \times (3d-1)$ and the number of control coefficients is $(q+1)^3$, while the number of determinants (component of the coefficients) is $(d(d+1)^2)^3$, see Table 5.

Remark As compared (see Tables 3, 4, 5) with tetrahedra and prisms, hexahedra have a jacobian with a higher degree and, in turn, the number of control coefficients is largely much more high for a given degree, even more than the triangles versus the quadrilaterals in two dimensions.

In practice Given the nodes of the element, one should be able to define the control points. As for the previous elements, the control points corresponding to the edges are obtained using the formulae we have in two dimensions (see Appendix) and for those corresponding to a face, the solutions given for a quadrilateral apply. For the eventual internal control points, one has to solve the corresponding system.

7.8 Order d pyramids

We define these elements by the same way we used to define the 1-order pyramid, and therefore, we have

$$\theta(u,v,w) = \sum_{i=0,d}\sum_{j=0,d}\sum_{k=0,d} B_i^d(u)B_j^d(v)B_k^d(w)\,P_{ijk} \tag{42}$$

with $P_{ijd} = P_{00d}$ for all couple (i,j). Then

$$\theta(u,v,w) = \sum_{i=0,d}\sum_{j=0,d}\sum_{k=0,d-1} B_i^d(u)B_j^d(v)B_k^d(w)\,P_{ijk}$$
$$+ \sum_{i=0,d}\sum_{j=0,d} B_i^d(u)B_j^d(v)B_d^d(w)\,P_{00d},$$

and $\sum_{i=0,d}\sum_{j=0,d} B_i^d(u)B_j^d(v)$ factorizes so we have

$$\theta(u,v,w) = \sum_{i=0,d}\sum_{j=0,d}\sum_{k=0,d-1} B_i^d(u)B_j^d(v)B_k^d(w)\,P_{ijk}$$
$$+ B_d^d(w)\,P_{00d},$$

where $B_d^d(w) = w^d$, from which we can easily define the shape functions.

The jacobian polynomial is that of the order-d hexahedron but a number of terms is null so, as previously, we have

$$\mathcal{J}(u,v,w) = \sum_{I=0,q}\sum_{J=0,q}\sum_{K=0,q} B_I^q(u)B_J^q(v)B_K^q(w)N_{IJK},$$

where $q = 3d-1$ and the coefficients N_{IJK} are

$$N_{IJK} = d^3 \sum_{|i|=I}\sum_{|j|=J}\sum_{|k|=K} \frac{C_{i_1}^{d-1}C_{i_2}^{d}C_{i_3}^{d}}{C_I^q}\frac{C_{j_1}^{d}C_{j_2}^{d-1}C_{j_3}^{d}}{C_J^q}$$
$$\frac{C_{k_1}^{d}C_{k_2}^{d}C_{k_3}^{d-1}}{C_K^q}|\Delta_{i_1j_1k_1}^{100} \quad \Delta_{i_2j_2k_2}^{010} \quad \Delta_{i_3j_3k_3}^{001}|,$$
$$\text{with } \Delta_{ijk}^{100} = \overrightarrow{P_{ijk}P_{i+1,jk}}, \Delta_{ijk}^{010} = \overrightarrow{P_{ijk}P_{i,j+1,k}}$$
$$\text{and } \Delta_{ijk}^{001} = \overrightarrow{P_{ijk}P_{ij,k+1}}. \tag{43}$$

When $k = d$, $\Delta_{ijk}^{100} = \Delta_{ijk}^{010} = \overrightarrow{0}$. Since $k_1 = 0, d-1, k_2 = 0, d-1, k_3 = 0, d-1$ and $K = k_1 + k_2 + k_3$, the maximum in K is $3d-3$ and then $(1-w)^2$ factorizes and the Jacobian has a singularity at $w = 1$, $\mathcal{J}(u,v,1) = 0$, and, in fact, the jacobian reads

$$\mathcal{J}(u,v,w) = \sum_{I=0,q}\sum_{J=0,q}\sum_{K=0,q'} B_I^q(u)B_J^q(v)B_K^q(w)N_{IJK},$$

where $q = 3d-1$ but with $q' = 3d-3$, meaning that some coefficients are missing.

The jacobian is a polynomial of degree $q \times q \times q' = (3d-1) \times (3d-1) \times (3d-3)$ and the number of control coefficients (apart for the degree 1 which is much more

Table 6 Statistics about the pyramids of degree 1 to 5

d	$\#nodes$	$q \times q \times q$	$\#coef$	$\#terms$
1	5	$2 \times 2 \times 2$	5	8
2	19	$5 \times 5 \times 5$	144	2,592
3	49	$8 \times 8 \times 8$	350	62,208
4	101	$11 \times 11 \times 11$	1043	640,000
5	281	$14 \times 14 \times 14$	2294	4,050,000

simple) is derived from the number we had for the hexahedron, therefore $(q+1)^3 - 2(q+1)^2$ while the number of determinants (component of the coefficients) is $d^5(d+1)^4$, see Table 6.

Remark As compared with an hexahedron, we have a small gain in terms of computing cost.

In practice Given the nodes of the element, one should be able to define the control points. The method used for the hexahedra gives the solutions.

8 Tridimensional incomplete Lagrange elements

First of all, formulae (28) and (29) extend to hexahedra and simplices and show that the reduced shape functions are related to the complete shape functions. Then, some of the complete Lagrange elements have their related incomplete elements. The method to constructing those reduced elements is basically the same, e.g. by means of Taylor expansions while taking into account what polynomial space we like to have.

Just to give two examples of reduced elements, we consider the case of a tetrahedron of degree 3 and the well-known hexahedron of degree 2.

The complete simplex of degree 3 has 20 nodes, and there is one node by face (and no internal node). The restriction to such a face is a triangle of degree 3 and the way to obtain a reduced simplex is to apply what we did for the bidimensional triangle to the four faces. This results in an element with 16 nodes.

The complete hexahedron of degree 2 has 27 nodes, and there is one node by face and one node. The restriction to such a face is a quadrilateral of degree 2 and the way to obtain a reduced element is to apply what we did for the bidimensional quadrilateral to the six faces. This results in an element with 21 nodes where there is only one internal node which, in turn, can be suppressed so as to obtain a 20-node element. It could be observed that this element can also be obtained by means of a transfinite interpolation, [18].

The method to check the geometric validity is similar to that used in two dimensions, we *invent* the "missing" nodes and the "missing" control points prior to apply the classical analysis of the jacobian polynomial.

9 Conclusion

We have discussed the theoretical framework that makes it possible to evaluate the geometric validity of the Lagrange finite elements. The use of the Bézier reading of these elements gives us a sufficient condition of positivity, a refinement algorithm being used to make the evaluation more precise.

As seen in the paper, the practical aspects include two main technical questions. One the one hand, it is required to define the control coefficients (this relies to finding the constitutive determinants or, in other words, to finding the appropriate indices of the end-points of the vectors involved in these determinants) and, on the other hand, the number of control coefficients (and the number of determinants to be computed) rapidly increases with the degree of the elements under analysis, thus leading to a problem of computing cost.

To conclude, note that the above theory does not directly apply to surface triangles and quads since the transformation is from R^2 to R^3 and the notion of a jacobian is not well defined in such a frame. A proper solution could be to define virtual elements in three dimensions with such triangles or quads as a face and to apply the evaluation method, now well defined, to the restriction of the jacobian polynomial to the relevant (tridimensional) faces.

Future investigations naturally include a proper definition of high-order elements shape quality. We think that measures using only Jacobians (for example, the ratio of the min and the max), i.e. volumes are not sufficient to quantify the regularity of an element. In [19], a measure of the distortion of an element with respect to its straight-sided counterpart is introduced before being combined with the shape quality of the latter thus giving a first method where the shape is really taken into account.

Appendix

Edge nodes versus control points

In this Appendix, we give some tables where it is shown, for the first orders (from 1 to 5), how the edge nodes are related to the edge control points and, conversely, how the edge control points can be found when the data are made up of the edge nodes (as it is in the finite element world).

Tables 7 and 8 depict the case of curved edges where natural indices are used (as it is for quads, quadrilateral

Table 7 Complete or Serendipity Lagrange quadrilaterals of degree 2 to 5

d	Edge nodes in terms of control points
2	$A_{10} = \frac{P_{00}+2P_{10}+P_{20}}{4}$
3	$A_{10} = \frac{8P_{00}+12P_{10}+6P_{20}+P_{30}}{27}$
	$A_{20} = \frac{P_{00}+6P_{10}+12P_{20}+8P_{30}}{27}$
4	$A_{10} = \frac{81A_{00}+108P_{10}+54P_{20}+12P_{30}+A_{40}}{256}$
	$A_{20} = \frac{A_{00}+4P_{10}+6P_{20}+4P_{30}+A_{40}}{16}$
	$A_{30} = \frac{A_{00}+12P_{10}+54P_{20}+108P_{30}+81A_{40}}{256}$
5	$A_{10} = \frac{1024A_{00}+1280P_{10}+640P_{20}+160P_{30}+20P_{40}+A_{50}}{3125}$
	$A_{20} = \frac{243A_{00}+810P_{10}+1080P_{20}+720P_{30}+240P_{40}+32A_{50}}{3125}$
	$A_{30} = \frac{32A_{00}+240P_{10}+720P_{20}+1080P_{30}+810P_{40}+243A_{50}}{3125}$
	$A_{40} = \frac{A_{00}+20P_{10}+160P_{20}+640P_{30}+1280P_{40}+1024A_{50}}{3125}$

Table 8 Complete or Serendipity Lagrange quadrilaterals of degree 2 to 5

d	Edge control points in terms of nodes
2	$P_{10} = \frac{-A_{00}+4A_{10}-A_{20}}{2}$
3	$P_{10} = \frac{-5A_{00}+18A_{10}-9A_{20}+2A_{30}}{6}$
	$P_{20} = \frac{2A_{00}-9A_{10}+18A_{20}-5A_{30}}{6}$
4	$P_{10} = \frac{-13A_{00}+48A_{10}-36A_{20}+16A_{30}-3A_{40}}{12}$
	$P_{20} = \frac{13A_{00}-64A_{10}+120A_{20}-64A_{30}+13A_{40}}{18}$
	$P_{30} = \frac{-3A_{00}+16A_{10}-36A_{20}+48A_{30}-13A_{40}}{12}$
5	$P_{10} = \frac{-77A_{00}+300A_{10}-300A_{20}+200A_{30}-75A_{40}+12A_{50}}{60}$
	$P_{20} = \frac{269A_{00}-1450A_{10}+2950A_{20}-2300A_{30}+925A_{40}-154A_{50}}{240}$
	$P_{30} = \frac{-154A_{00}+925A_{10}-2300A_{20}+2950A_{30}-1450A_{40}+269A_{50}}{240}$
	$P_{40} = \frac{12A_{00}-75A_{10}+200A_{20}-300A_{30}+300A_{40}-77A_{50}}{60}$

faces, hexes, ...). Tables 9 and 10 are identical but here we used the indices as they are defined in a barycentric system (as it is for triangle, triangular faces and tetrahedra), and therefore, we only give the degree 3. The relation from one system to the other is as follows (here for the degree 3):

00	10	20	30
300	210	120	030

Since complete elements and reduced elements have the same boundary edges, the relations hold for both cases.

In the tables, we consider the case of bidimensional edges but, obviously, this apply to three dimensions after an adequate labelling of the entities.

Table 9 Complete or reduced Lagrange triangle of degree 3

d	Edge nodes in terms of the control points
3	$A_{210} = \frac{8P_{300}+12P_{210}+6P_{120}+P_{030}}{27}$
	$A_{120} = \frac{P_{300}+6P_{210}+12P_{120}+8P_{030}}{27}$

Table 10 Complete or reduced Lagrange triangle of degree 3

d	Edge control points in terms of the nodes
3	$P_{210} = \frac{-5A_{300}+18A_{210}-9A_{120}+2A_{030}}{6}$
	$P_{120} = \frac{2A_{300}-9A_{210}+18A_{120}-5A_{030}}{6}$

Determinants in a control coefficient

We explain how to find the determinants involved in one control coefficient by considering the case of a quad of degree d, Relation (27).

At first, it is required to find the appropriate indices i_1 and i_2 such that ($q = 2d - 1$)

– (For $I = 0, q$, (for $i_1 = 0, d-1$ (for $i_2 = 0, d$ $(i_1 + i_2 = I)$))),

from which we have the pairs (i_1, i_2) therefore the pertinent vectors together with the associated weights (the factors $C_{..}^{..}$) relative to these indices. Then we do the same for the indices j_1 and j_2 so we find the pairs such that

– (For $J = 0, q$, (for $j_1 = 0, d$ (for $j_2 = 0, d-1$ $(j_1 + j_2 = J)$))),

so we have the pairs (j_1, j_2) therefore the pertinent vectors together with the associated weights (the factors $C_{..}^{..}$) relative to these indices. Now, we group together these indices to obtain the pairs (i_1, j_1) and (i_2, j_2), the weights and the number of terms (determinants) in a given coefficient. In Table 11 and the following schematics, we give details about the coefficients N_{IJ} when $J = 0$, note that $vector_1$ span the edge $P_{00}P_{d0}$, in the u-direction, from $\overrightarrow{P_{00}P_{10}}$ to $\overrightarrow{P_{I0}P_{I+1,0}}$ (top to bottom) while $vector_2$ are the vectors in the v-direction from $\overrightarrow{P_{00}P_{01}}$ to $\overrightarrow{P_{I0}P_{I1}}$ (bottom to top). For $J = 1$, we have to repeat the same task but now we have two pairs in (j_1, j_2), one being (0, 1), and the other being (1, 0), Table 12, increasing the number of determinants in a given coefficient, etc. Finding all the coefficients clearly requires writing a computer program.

Table 11 Example of how to compute some control coefficients when $J = 0$

IJ	i_1	i_2	j_1	j_2	$vector_1$	$vector_2$	Comment
00	0	0	0	0	$\overrightarrow{P_{00}P_{10}}$	$\overrightarrow{P_{00}P_{01}}$	corner
10	0	1	0	0	$\overrightarrow{P_{00}P_{10}}$	$\overrightarrow{P_{10}P_{11}}$	corner $d = 1$
	1	0	0	0	$\overrightarrow{P_{10}P_{20}}$	$\overrightarrow{P_{00}P_{01}}$	$d > 1$
20	0	2	0	0	$\overrightarrow{P_{00}P_{10}}$	$\overrightarrow{P_{20}P_{21}}$	
	1	1	0	0	$\overrightarrow{P_{10}P_{20}}$	$\overrightarrow{P_{10}P_{11}}$	
	2	0	0	0	$\overrightarrow{P_{20}P_{30}}$	$\overrightarrow{P_{00}P_{01}}$	$d > 2$
30	0	3	0	0	$\overrightarrow{P_{00}P_{10}}$	$\overrightarrow{P_{30}P_{31}}$	
	1	2	0	0	$\overrightarrow{P_{10}P_{20}}$	$\overrightarrow{P_{20}P_{21}}$	corner $d = 2$
	2	1	0	0	$\overrightarrow{P_{20}P_{30}}$	$\overrightarrow{P_{10}P_{11}}$	
	3	0	0	0	$\overrightarrow{P_{30}P_{40}}$	$\overrightarrow{P_{00}P_{01}}$	$d > 3$
40	0	4	0	0	$\overrightarrow{P_{00}P_{10}}$	$\overrightarrow{P_{40}P_{41}}$	
	1	3	0	0	$\overrightarrow{P_{10}P_{20}}$	$\overrightarrow{P_{30}P_{31}}$	
	2	2	0	0	$\overrightarrow{P_{20}P_{30}}$	$\overrightarrow{P_{20}P_{21}}$	
	3	1	0	0	$\overrightarrow{P_{30}P_{40}}$	$\overrightarrow{P_{10}P_{11}}$	
	4	0	0	0	$\overrightarrow{P_{40}P_{50}}$	$\overrightarrow{P_{00}P_{01}}$	$d > 4$
50	0	5	0	0	$\overrightarrow{P_{00}P_{10}}$	$\overrightarrow{P_{50}P_{51}}$	
	1	4	0	0	$\overrightarrow{P_{10}P_{20}}$	$\overrightarrow{P_{40}P_{41}}$	
	2	3	0	0	$\overrightarrow{P_{20}P_{30}}$	$\overrightarrow{P_{30}P_{31}}$	corner $d = 3$
	3	2	0	0	$\overrightarrow{P_{30}P_{40}}$	$\overrightarrow{P_{20}P_{21}}$	
	4	1	0	0	$\overrightarrow{P_{40}P_{50}}$	$\overrightarrow{P_{10}P_{11}}$	
	5	0	0	0	$\overrightarrow{P_{50}P_{60}}$	$\overrightarrow{P_{00}P_{01}}$	$d > 5$
etc.							

```
degree 1     01       11          01       11



             00       10          00       10

                       03       13      23      33
degree 2

     Pij vs Nij        02       12      22      32

02      12      22
                       01       11      21      31
01      11      21



00      10      20     00       10      20      30
```

Table 12 Example of how to compute some control coefficients when $J = 1$

IJ	i_1	i_2	j_1	j_2	$vector_1$	$vector_2$	Comment
01	0	0	0	1	$\overrightarrow{P_{00}P_{10}}$	$\overrightarrow{P_{01}P_{02}}$	$d > 1$
	0	0	1	0	$\overrightarrow{P_{01}P_{11}}$	$\overrightarrow{P_{00}P_{01}}$	
11	0	1	0	1	$\overrightarrow{P_{00}P_{10}}$	$\overrightarrow{P_{11}P_{12}}$	$d > 1$
	0	1	1	0	$\overrightarrow{P_{01}P_{11}}$	$\overrightarrow{P_{10}P_{11}}$	
	1	0	0	1	$\overrightarrow{P_{10}P_{20}}$	$\overrightarrow{P_{01}P_{02}}$	$d > 1$
	1	0	1	0	$\overrightarrow{P_{11}P_{21}}$	$\overrightarrow{P_{00}P_{01}}$	$d > 1$
21	0	2	0	1	$\overrightarrow{P_{00}P_{10}}$	$\overrightarrow{P_{21}P_{22}}$	
	0	2	1	0	$\overrightarrow{P_{01}P_{11}}$	$\overrightarrow{P_{20}P_{21}}$	
	1	1	0	1	$\overrightarrow{P_{10}P_{20}}$	$\overrightarrow{P_{11}P_{12}}$	
	1	1	1	0	$\overrightarrow{P_{11}P_{21}}$	$\overrightarrow{P_{10}P_{11}}$	
	2	0	0	1	$\overrightarrow{P_{20}P_{30}}$	$\overrightarrow{P_{01}P_{02}}$	$d > 2$
	2	0	1	0	$\overrightarrow{P_{21}P_{31}}$	$\overrightarrow{P_{00}P_{01}}$	$d > 2$
31	0	3	0	1	$\overrightarrow{P_{00}P_{10}}$	$\overrightarrow{P_{31}P_{32}}$	
	0	3	1	0	$\overrightarrow{P_{01}P_{11}}$	$\overrightarrow{P_{30}P_{31}}$	
	1	2	0	1	$\overrightarrow{P_{10}P_{20}}$	$\overrightarrow{P_{21}P_{22}}$	
	1	2	1	0	$\overrightarrow{P_{11}P_{21}}$	$\overrightarrow{P_{20}P_{21}}$	
	2	1	0	1	$\overrightarrow{P_{20}P_{30}}$	$\overrightarrow{P_{11}P_{12}}$	
	2	1	1	0	$\overrightarrow{P_{21}P_{31}}$	$\overrightarrow{P_{10}P_{11}}$	
	3	0	0	1	$\overrightarrow{P_{30}P_{40}}$	$\overrightarrow{P_{01}P_{02}}$	$d > 3$
	3	0	1	0	$\overrightarrow{P_{31}P_{41}}$	$\overrightarrow{P_{00}P_{01}}$	$d > 3$
etc.							

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